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ABSTRACT

This set of five reports and one manual is a product of the Mathemagenic Activities Program (MAP) for early childhood education of the University of Georgia Follow Through Program. Based on Piagetian theory, the MAP provides sequentially structured sets of curriculum materials and processes that are designed to continually challenge children in kindergarten and the first three grades who are from low income families to learn about the physical and social environments. These papers focus on aspects of mathematics instruction. A model for learning and teaching mathematics, Piaget's thinking about the development of space concepts and geometry, needed research on space concepts in the context of the Geneva research group, psychological principles for mathematical instruction, and elementary and middle school children's comprehension of Euclidian transformations are discussed individually in five reports. The sixth paper is a teacher strategy manual for mathematics instruction. In seven chapters, this manual overviews the MAP, indicates MAP rationale and principles, discusses activity teaching for active learning, provides suggestions for implementing active learning in the classroom, suggests objectives and activities for grades K - 3, explores a model of learning and instruction in mathematics, and discusses the child's conception of number. (Author/RH)

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MATHEMAGENIC ACTIVITIES PROGRAM: REPORTS ON TEACHING AND LEARNING MATHEMATICS. (Title Supplied)

Charles D. Smock, Editor

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DISCOVERING PSYCHOLOGICAL PRINCIPLES
FOR MATHEMATICS INSTRUCTION

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DRAFT

Discovering Psychological Principles for Mathematics Instruction[1, 2]

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> It is not that they can't see the solution -- they can't see the problem.
>
> (Chesterton)

> Science is not just a collection of laws, a catalogue of unrelated facts. It is a creation of the human mind, with its freely invented ideas and concepts . . . The only justification for our mental structures is whether and in what way our theories form . . . a link with the world of sense impressions.
>
> (Einstein)

> The drive to lead an intellectually satisfying life is . . . a product of long process of education . . . (and) . . . is an autocatalytic affair, growing with the practice of it.
>
> (Bridgeman)

Currently, either the name Piaget or the term mathematics laboratory is sure to attract the interest of mathematics educators. While laboratories are gaining in fad appeal, precise meaning for such an instructional technique has remained only loosely and ambiguously defined. Teachers who have attempted to use such a teaching strategy have typically been forced

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to rely on a set of "rule of thumb" slogans, none of which has been shown to be valid in all learning situations.

"Concrete understanding before abstract;" "Intuitive understanding before formalization;" "Use activities, then pictures, them symbols;" "Use discovery rather than reception methods." All of these slogans refer to distinct instructional variables which occasionally specify contradictory approaches to teaching if their range of appropriateness is not qualified and coordinated by at least an embryonic theory of laboratory instruction.

Mathematics educators appear to have little interest in theory construction and have a restricted view of the role of a mathematics laboratory. Theory is used to justify instructional biases and the laboratory is considered only an instructional device. Theory construction can help clarify the different types of roles a laboratory has in mathematics education research and practice. The mathematics laboratory should be a context for research on problems relevant to specific aspects of the instructional process (variation of teaching strategies and techniques) and on discovering those conditions critical for mathematical learning, and development of mathematical thinking in children, as well as fun and games for child and teacher. If educational researchers expect to accumulate knowledge relevant to theory based instructional practices, they must have "mathematics" laboratories.

However, in my opinion, Chesterton's remark is quite appropriate: "It is not that they can't see the solution -- they can't see the problem." We have yet to identify the fundamental dimensions of educational and instructional problems facing the mathematics educator. Key psychological principles for mathematics instruction and construction of a theory of instruction can be realized only after this frist step has been achieved. A body of knowledge now exists in developmental-cognitive psychology that should have considerable utility for contributing to more refined theories of and strategies for instruction; i.e., for creating better school learning environments. One approach to the immediate task, then, is to search for suggestions from developmental psychology that pose relevant problems for mathematics educators for developing theories of mathematics instruction.

There is no paucity of choices of psychological models available from which to start the search. Berlyne, Bandura, Bower, Scandura, Skinner, Suppes -- and of course Piaget. Each has proposed a set of ideas (relevant to a theory of mathematics instruction) requiring theoretical and empirical study. The selection of any one model, however, brings with it many hidden presuppositions and is determined in no little part by one's own preconceived notion regarding human development, learning and education. As a group, these models represent a virtual wonderland of exciting ideas. Each of us can understand Alice's dilemma better as we explore their fantasies (Suppes, 1971).

A theory of instruction must begin with an adequate theory of learning and/or cognitive development. No longer can we accept that statement as "obvious" -- and go about the business of generating a multitude of methods based on unorganized intuitive rules constructed on the basis of inadequate knowledge of the process of cognitive development of children. Mathematics educators need to return to the beginning and ask not "how do we teach?" but rather "how do children learn?"

Modern developmental psychology provides a necessary, but not sufficient, body of knowledge for identifying some of the fundamental issues, constraints, and facts associated with the process of generating a theory of mathematics learning and instruction. But, to imply, and act as if, psychology has become relevant to mathematics learnings only AP (i.e., after Piaget), misses a fundamental point about the relation of the science of psychology to the science of education, and distorts the history of both. On the other hand, I do not want Piaget's theory of cognitive development abandoned without clear understanding of why. The historical pattern in education and psychology seems to be one of enthusiastic adherence to a relatively novel theory -- with disappointment and rejection following close behind. The absence of serious controversial issues underlying much of the current research in cognitive development, and mathematics learning, increases my concern that much that is valuable in Piaget's theory may be lost. This symposium, perhaps, will provide a more relevant set of issues about which we can disagree than have surfaced to this point.

Many psychologists, including myself, consider Piaget's clarification of the necessary bases of theory construction as important as his cognitive developmental theory *per se*. Implicitly, and explicitly, Piaget was greatly influenced by advances in theoretical physics (Bridgeman, 1929) during the 1920's and 30's. The fundamental aspect of relativity theory which cannot be ignored in psychological theorizing is that conceptual judgments are always relative to the position of the observer. Analysis of knowledge acquisition requires a description of the operational basis of these acquisitions; i.e., the mental operations of the individual that are associated with the construction and maintenance of consistent patterns (structure) of his continually transforming relations with the physical and social environments. Thus, Piaget's emphasis on a constructivistic theory of knowledge (Piaget, 1968, 1971) is indissoluble from his interpretation of operationism -- that is, the need for operational analysis of the process of knowledge acquisition. "Reality" is constructed, not eminent in mind, man or stimulus, and applies to the child and, contrary to some interpretations (e.g., Stevens, 1935), theorist alike.

The form of epistemology typical of American psychologists (cf: Mischel, 1971) has been naive realism, and that orientation has been quite useful. We need not abandon it completely, but, it should now be obvious that our epistemological preconceptions, whatever they may be, are part of our view of the child. Kessen (1966) states the issue clearly: "The child who is confronted by a stable reality that can be described adequately in the language of contemporary physics, is a child very different from the one who is seen facing phenomenal disorder from which he must construct a coherent view of reality" (pp. 58-59).

Analysis of cognitive learning and development, then, is always "biased" by the fact of a context of preconceived ideas of reality (i.e., western culture) and a particular set of concepts or theory and selected data. Piaget's approach to the analysis of the development of children's conception of space provides us an excellent example (Piaget, J. and Inhelder, B., 1956). The designation of a conception of space toward which the child will develop, i.e., that conception held by the educated adult, is the critical first step. Once the endpoint for this development has been stated, observations and interpretation of the child's behavior are organized around these specifications. Observations not congruent with, and not structurable in terms of the specified endpoints, no matter how reliable, cannot be considered a relevant part of development. This is not an example of bad science or inappropriate procedures but rather illustrates that conclusions about the child's cognitive apparatus may be based as much on the construction of reality imposed on him in the first place as on the reliability and generality of the observations we make. Whether we are engaged in instructional practices, or instructional research, or theory building, there is, for each of us, a set of guiding propositions that constitutes a theory of learning and development. These "fantasies" or "freely invented ideas and concepts" provide a particular coherent view of the developing child and of the critical determinants of the learning process.

Piaget's epistemology, and his biological background, thus predispose him toward an operational and structural analysis of the knowledge acquisition process (Piaget, 1967; 1968; 1970; 1971). The essentials of his position require only brief review there. Knowledge is defined as invariance under transformation (a most familiar concept to mathematicians). The construction of invariances in organism-environment relations takes place through the operation of two complimentary biological adaptation processes, both of which are under the control of the internal self-regulating mechanism of equilibration.

One of the two processes (assimilation) concerns the application of existing cognitive operational systems (structures) to the processing of environmental (sensory) data. External data or events are incorporated into existing structures through both on-going physical and mental activity. Such events and the products of new experience can be incorporated into the cognitive structure only to the extent it is consistent with existing functional structures.

Accommodation is the complementary process whereby adaptation occurs by integration of existing structures with other functional structures and/or by differentiation of new structures under confrontation with new experience.

Activities such as play, practical or symbolic, represent assimilative activity; whereas memory, in the sense of invoking past events, and imitation are accommodative since only existing structures are brought to bear on particular events or set of events. Assimilation is an active constructive process by which the data from experience are transformed and integrated with an already generalized cognitive structure. Accommodative activity, on the other hand, is associated with the process whereby application of existing structures are brought to bear on particular new events or sets of events, i.e., events or sets of events to which these structures have not been applied previously.

Too often instructional theory and practice have emphasized assimilation (i.e., "play") or accommodation (i.e., imitation) activity and neglected the role of equilibration of these complementary processes for cognitive learning. Appropriate generalization of Piaget's ideas to instructional theory and practice requires consideration of three additional factors associated with Piagetian theory: "logic," operative vs. figurative thought, and equilibration.

Operational Structures and "Logic"

Piaget (1970) recently elaborated his position that human beings possess the same biological structures and functions that, in "exchange" with the common features of the natural world, generate mental (operational) structures and functions characteristic of each stage of development. Logical thought, in the Piagetian sense, then, is universal and of fundamental importance to an understanding of development and learning. But, whereas Chomsky maintains the human mind is "programmed" at birth with cognitive structures (i.e., mental representation of a universal grammar), Piaget accounts for the universality and stability of structures across cultures (Piaget, 1967; Goodnow, J., 1962; Goodnow, J. and Bethon, G., 1966; Greenfield, 1966; Maccoby, M. and Modiano, N., 1966) in terms of the self-regulation mechanism of equilibration. Thus, Piaget proposes that the mind at any point in development (i.e., life) is the unfinished product of continual self-construction (1971); i.e., "logical" processes are generative and not fixed. Structures are not preformed, but are self-regulatory, transformational systems with the functional factors in that construction being the processes of assimilation and accommodation.

Intelligence, the basis for knowledge acquisition, consists, then, of two aspects: adaptation (with the complementary processes of assimilation-accommodation and the self-regulatory mechanism of equilibration) and organization consisting of sets of mental operations that form the basis for maintaining invariance under transformation (i.e., knowldege). It follows from these considerations that there is an inherent logic to development; i.e., operational systems consist of elements and laws of combination of those elements that from a "logical" closed system. These mental structures are observable in the actions of the organism in its environment and, further, are describable in terms of formal or logico-mathematical structures. Genetic psychological analysis of these structures is a necessary prerequisite to an understanding of thought processes. "No structure without genesis, no genesis without structure" (Piaget, 1968). During the sensorimotor period of development, action structures of the individual are revealed in practical groups, i.e., the observable coordinated actions of the individual (Forman, 1973). During the pre-operational period, the child constructs representations (figurative structures) which do not have the operational property of reversibility. Piaget was able to identify operational structures with clear mathematical system properties in children between ages five and seven. The discovery of a resemblance between the structure of the mental action system (reasoning or thought) and mathematical structures (i.e., mathematical groups and lattices) had a profound effect on Piaget's thinking. Thought, it would appear, has the same, or similar properties, as mathematical group structures, both of which are governed by the same internal logic.[3]

The basic structuralistic approach of Piaget involves finding logical mathematical systems that describe the thought processes of an individual. Mathematical group and lattice theory offers algebraic systems (Piaget, 1957; Flavell, 1963) for describing operational thought and Piaget has tried to take maximum advantage of that approach. To review the fundamentals: a mathematical group is a system consisting of a set of elements, together with an operation (law of combination), which yields the following system properties: (1) when applied to the elements of the set, the combinatorial operation will yield only element of the set; no elements external to the set can be produced; (2) each set contains a neutral (or identity) element that when combined with any other element of the set yields no change; (3) each element of the set has an inverse which in combination with any element yields the neutral or identity element; (4) the combinatory operation (and its inverse) is associative; i.e., $[(n + m) + p = n + (m + p)]$. Piaget found it necessary to generate a "grouping" model with additional properties (i.e., both group and lattice properties) to describe the concrete operational structures. The properties of these *psychological groupings* are not derived from the properties of things, but from *modes of acting upon things*. Thus, the elements of psychological groups are, themselves, transformations that characterize the individual's operations as he acts upon incoming sense data.

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[3] Piaget never has tried to find a mathematical "model" to "fit" the observed facts of behavior; rather the mathematical aspect of Piaget's theory is unique in that he assumes, somewhat a reminescent of Boole's "laws of thought," an identity between the inherent logic of thought processes and certain formal mathematical systems that have become "externalized," through inductive reasoning, and guide the action-patterns of the individual.

The revelations emerging from relativity theory require a constructionistic position with respect to the nature of knowledge; i.e., understanding of knowledge acquisition requires a description and characterization of the mental operations and operational systems applied to the data of experience. Piaget's emphasis on structural analysis thus makes contact with the epistemological implications emerging from relativity theory; the biologists' emphasis on development as the formation, differentiation and hierarchical integration of functional (action) structures; and the mathematicians' emphasis on formalized systems that permit description of these structures. The task of the developmental psychologist is to describe the nature of action structures of the child at each point in development and, as much as possible, to formalize those descriptions in terms of logico-mathematical terms.

The classical "conservation" tasks, if administered appropriately, form one basis for generating observation of coordinated actions that appear to reflect such mental operational structures. The available evidence appears to support the possibility that such operational (mental) structures "exist" both in terms of replicability of developmental trends in task solution and training studies (cf: Beilin, 1971a). At the same time, neither psychological nor educational researchers have yet devoted sufficient attention to the problems of the validity (i.e., internal consistency) of the grouping structures (Clary, 1970; Green, D., 1971), nor to the role of such structures in learning (Berlyne, 1961; Bruce, 1971; Inhelder, B. and Sinclair, H., 1969), beyond these classical situations.

Only recently have mathematics educators become interested in Piaget's views of fundamental logico-mathematical relations, such as his ideas about the logical properties of number and space. Beilin (1971b) points out that philosophers of science generally have emphasized the desirability of isolating philosophical and logical systems from psycho-logical matters. Psychologists, mathematicians and logicians generally have maintained this position with respect to Piaget. However, a significant part of his psychological theory has mathematical and logical content which cannot be ignored (Leskow, S. and Smock, C. D., 1970; Alonzo, M., 1967) by either psychological or mathematics learning researchers. Mathematics educators rightly should be directed in part, to the analysis of the logical and mathematical veracity of Piaget's system and to the correspondence between the characteristics of the psycho-logic systems to those logic structures derived from purely mathematical analysis. Recent work from Steffe's laboratory (e.g., Kidder, 1973; Johnson, D., 1971; Lesh, R., 1971; Johnson, M., 1971) represents an excellent beginning in this direction.

Role of Experience and Equilibration

Experience is not a unique factor in development according to Piaget. Merely being exposed to particular experiences is conducive neither to cognitive activity nor to developmental change. Children may or may not make discoveries in the course of play, and watching a laboratory experiment (or conducting one) may or may not help a child acquire a particular concept. Equilibration (Langer, J., 1973) is the central factor in structural changes whether the reference is to stage or concept learning. Equilibration is the process of intrinsic (self) regulation that balances assimilatory and accommodative processes, compensates for external and internal disturbances and makes possible the development of more complex, hierarchically integrated operational structures. Disequilibrium occurs as the child assimilates data from exchanges with the environment into existing mental structures. As cognitive structures change to accommodate to the new information, equilibrium is restored. The equilibration process is one of auto-regulation -- both of the transformations of information based on existing cognitive structures and of changes through accommodation. In any case, the child must be exposed to environmental input that "engages" the functional structures; i.e., he must be involved in a personal striving to understand or "accept" the task as a "problem."

A basic question for instructional theory and practice is: What are the processes and conditions that motivate and insure engagement or acceptance of the problem task by the child? The source of "interest" that promotes striving for problem solution is contingent on assimilative-accommodative activity but the specifics remain unclarified in Piaget's theory (cf: Mischel, 1972). What Piaget means by the "need" of structures to function is a "primitive" factor not unlike the notion of competence "drive" suggested by White (1969). Within a structuralist framework -- if a structure exists, it must function -- cognitive structures appear to have a dimension of openness that make probable continual sources of disequilibrium from interaction of the internal operational and/or figurative structures activated as well as by exchanges involving novel environmental input. In any case, natural or lifelike contexts seem to provide excellent situations in terms of promoting cognitive change. Despite lack of specifications, Piaget is quite explicit on his position: "It is not necessary for us to have recourse to separate factors of motivation in order to explain learning, not because they don't intervene... but because they are included from the start in the concept of assimilation... to say that the subject is interested in a certain result or object thus means that he assimilates it or anticipates an assimilation and to say that he needs it means that he possesses schemas requiring its utilization" (Piaget, 1959).

Cognitive conflict, or the awareness of a momentary disequilibrium, represents a need to establish consistency (equilibrium) between the existing schemas and/or novel information and is motivation for cognitive activities. Both applying an existing schema and elaborating new ones in the course of development stem from simply the overriding need to make "sense" of present problems by fitting them coherently into schemas "learned" in the course of solving prior problems.

The notion that disturbances introduced into the child's systems of prior schemas lead to the adoption of a strategy for information processing is the fundamental difference between the equilibration and associationistic theories of learning (Piaget, 1957b). For associationistic theories of learning, "what is learned" depends on what is given from the outside (copy theory) and the motive that facilitates learning is an inner state of some sort of other. Equilibration theory holds, however, that learning is subservient to development; i.e., what is learned depends on what the learner can take from the given by means of the available cognitive structures. Further, cognitive disequilibrium (functional need) is what motivates learning (i.e., questions or felt lacunae arising from attempts to apply schemas to a "given" situation).

The child then will take interest in what generates cognitive conflict; i.e., in what is conceived as an anomaly. If the task demands are too novel as to be unassimilable or so obvious as to require no mental work, the child will not be motivated.

After the period of sensori-motor development, equilibration becomes a process of compensating for "virtual" rather than actual disturbances. At the operational level, intrusions "can be imagined and anticipated by the subject in the form of the direct operations of the system -- the compensatory activities will also consist of imagining and anticipating the transformations to an inverse sense" (Piaget, 1967. Further, there need be no external intrusions in order for the equilibration process to be activated. For example, the acquisition of conservation concepts is, in Piaget's view, "not supported by anything from the point of view of possible measurement or perception -- it is enforced by logical structuring much more than by experience" (Piaget, 1967). It is the "internal factors of coherence -- the deductive activity of the subject himself" that is primary. Equilibration, we noted earlier, is a response to internal conflict between conceptual schemas rather than a direct response to the character of outside structure factors. Equilibration is a matter of achieving "accord of thought with itself" in the service of establishing accord of thought with things.

Unfortunately, little empirical investigation has been oriented to questions of situational determinants of curiosity (Smock, C. D. and Holt, B.G., 1962) of children at various stages of development and with different experiential backgrounds; i.e., what children recognize as problematic, and what kinds of incongruities are sufficient to motivate change in concepts and/or beliefs.

Operative and Figurative Thought

A considerable amount of confusion concerning Piagetian theory and its implication for both research and instructional practice derives from a failure to consider the figurative and operative aspect of intellectual functioning. In general terms, the distinction is between the selection, storage and retrieval, and the coordination and transformation of information (Inhelder, et.al., 1966). More specifically, the development of any sequence of psychological stages, a la Piaget, consists of an interactive process of equilibrating functional structures of the organism with the event-structures of the environment. Figurative and operational processes represent two types of functional structures necessary to account for knowledge acquisition, development and learning. Figurations are defined as those action schemata that apprehend, extract and/or reproduce aspects of the physical and social environment. Such action schemata include components of perception, speech, imagery, and memory. Figurations and associated acts are based on physical, as contrasted to logico-mathematical, experience and constitute the "empirical" world; i.e., empirical truth is no more than the "representation of past events in memory."

Operations, on the other hand, do not derive from abstractions from objects and specific events; rather, operational knowledge is derived by abstractions from coordinated actions on those events. Thus, operations are those action schemata that construct "logical" transformations of "states." Such logical systems of transformations operate either upon representations of events, or on the cognitive system's own logical operations, i.e., reflexive operations.

Figurative and operative structures are two parallel streams with their genetic or developmental origins in the same source (Piaget, 1967, 1968; Piaget, 1970, 1971) -- the sensori-motor structures. Logical (operational) structures are not generated from the figurative schemata, i.e., not from perception, memory, etc. Reciprocally, figurative structures do not derive from operative schemas but from the representations of past states of events derived from physical experience. And most importantly, figurative structures do not derive from each other, but have unique bases in sensori-motor schema. Imagery, for example, is a derivative of deferred sensori-motor or imitation (Piaget, 1951; 1952; 1971) and not perception.

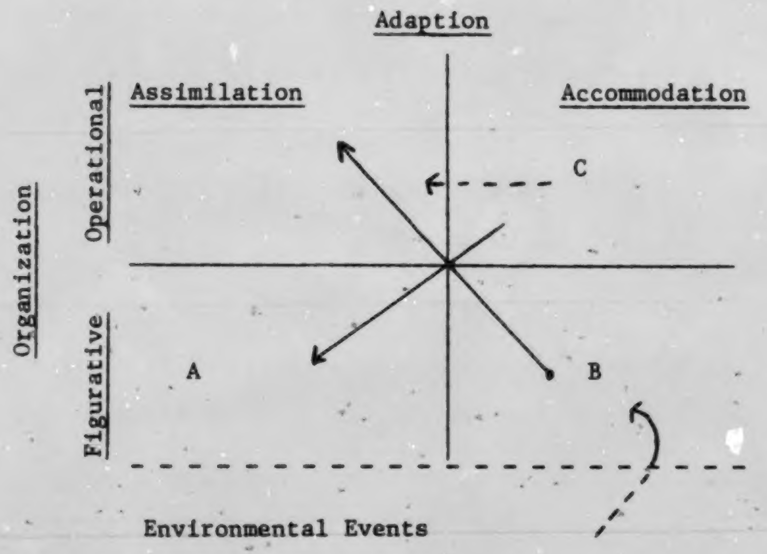
The postulation of these quite distinct functional structures is one of the cornerstones of Piaget's theory of knowledge acquisition and cognitive development (cf: Furth, 1969). Both the source and function of the structures are theoretically distinct. Operative structures derive from abstraction from coordinated actions, figurative structures derive from sensori-motor and perceptual activity. Operative structures produce "logical" transformations (conservation of invariants) whereas figurative structures reproduce sensory-perceptual consequences of environmental configurations. The variant operative structures of the intuitive, the concrete, and the formal levels form the discontinuous sequence of stages of cognitive development. On the other hand, figurative structures are static and dependent directly upon the data of experience (sensory-perceptual consequences of stimulation). Thus, Piaget makes the fundamental assumption that all knowledge acquisition activity is constructive, but the construction of figural representations is quite a distinct process from that constructive activity at the operative level.

Logically, there are three possible relations between the figurative and operational structures. First, they may be unrelated and, if so, as mentally segregated functional structures, do not set limits on the functioning and development of each other. Second, psychological phenomena might be reduced to one of the types of structures. Langer (1969) suggests that subjective idealists, perhaps, try to reduce psychological phenomena to assimilatory operations; and there are many theorists who try to reduce all mental phenomena to accommodatory figurations while naive realists propose that such processes are figurative; i.e., perception is knowledge (Michotte, 1963; Garner, 1962). Third, and the one proposed by Piaget, is that of partial communication between figurative and operative structure within the constraints of assimilation and accommodation processes.

The relations, and the potential form of interaction are schematically presented in Figure 1 below.

Figure 1

Relations of Two Invariant Processes of Adaptation and Two Types of Cognitive Structures



Langer (1969) has examined the organizational and developmental (i.e., transitional) impact of accommodatory figurations on assimilatory operations (See B, Figure 1); i.e., how does the child mentally extract and/or represent empirical information about physical and social objects and the consequences of that empirical activity for the construction of logical concepts. Imitation of an observed event, comparison of one's predictions with the outcome of a physical deformation, comparison of observation or appearance with the way things really are, represent different modes of introducing conflict and cognitive-structural change. Generally, his findings are confirmatory, but not definitive with respect to the Piagetian hypotheses. In any case, if the development of each type of functional structure has implications for, but not direct causal effects upon, the functional structure and development of the other, current paradigms for the study of learning mathematical concepts will require considerable modification. The work of the Geneva group mentioned earlier concerning, for example, memory (see A, Figure 1) and Langer's (1969b) analysis of the impact of accommodatory figurations (i.e., imitation, etc.) on assimilatory operations represents beginnings in this direciton.

Analysis of learning, in the context of Piagetian theory, poses requirements for much more detailed empirical analysis than has been generally recognized. On the one hand, researchers attempting to assimilate Piaget to their own conceptual structures concentrate on experimental procedures whereby the subject is required only to remember event contingencies (e.g., response-reward associations or "a" follows "b," follows "c"). Such procedures certainly produce change in "behavior" (e.g., Gellman, 1969; Mehler and Bever, 1967; Bever and Mehler, 1968); however, failure on transfer tasks, and a lack of persistence of task solution over time indicates that a figurative process underlies the change in performance. On the other hand, the accommodators (i.e., those more favorable toward Piaget's theory) often fail to generate experimental paradigms that adequately differentiate between the figurative and operational knowledge (Wallach, L. and Sprott, 1964) or assume that "external disparity" (appearance vs. "reality") is sufficient to establish disequilibrium (conflict) between logical necessity (derived from the operational structures) and perceptual pregnance (cf: Bruner, 1966). For example, situations designed to establish disparity between the child's predictive judgment of the outcome of a transformation and his observation of the actual outcome may, in fact, generate little or no cognitive conflict. Certainly, a most parsimonious explanation of many negative findings in training studies is that such disparity is external to the child's logical operational system.

Implications for Learning and Instruction

In some form or other, the goals of American educators have always been stated in terms of "optimizing" the intellectual, social, etc., development of individual children; a vague statement, obviously, and subject to a variety of interpretations. Whatever imperatives that goal implies, the educational and instructional processes must be based upon an understanding of the nature of psychological development of children. Whether we want to produce individuals who will strive to maintain the status quo; individuals who desire and accept change; people content to be technologists (i.e., skilled labor); or problem solvers; it is necessary to understand the basic processes of child development and the conditions that permit "quality control of the product" (if I may use current jargon).

The issue is important because science (i.e., theory and research) can only yield "what is" and not what "ought to be." We are fortunate, in one sense, that the science of psychology (and of pedagogy) is young and imperfect; the proposed models and methods for educating young children are no less imperfect and are influenced as strongly by current social thought and individual philosophical biases, as by an understanding of the laws of psychological development. Such a state of affairs, while producing wasted efforts, spurious claims, more rediscoveries that discoveries, etc., can at least provide time for the development of articulated sets of societal goals for education.

The best that can be hoped for, under the current conditions of our knowledge, is devleopment of preliminary "models" for instruction. Such models can provide, at least, a schematic set of principles and guidelines for constructing a learning environment consistent with the admittedly inadequate theories and knowledge of psychological growth. However, we should try not to violate recent advances in theory and known laws of child development.

Piaget (1961b) has rightfully declined to generalize his theory to specifics for educational practice. He has, however, suggested a theory of knowledge acquisition which has contributed to clarification and integration of a particular set of propositions concerning psychological development (many of which have a long history in child psychology and education). If we accept the fact that his theory of cognitive development is not yet a conceptually or empirically "closed system," several deductions concerning the construction of "optimum" environments can be generated. A modest attempt in this direction has been made at the University of Georgia-Follow Through Program (Smock, 1969). The initial stage of that model is based on an attempt to generate a set of instructional principles based on our understanding of Piaget's theory of cognitive development. Though many of the basic propositions upon which the model is based are not inconsistent with Piaget's thinking about knowledge acquisition, the interpretation is that of the modeler. It is influenced, therefore, by numerous sources of bias, misunderstanding, distortions, etc., that are inevitable under conditions where abstract theoretical concepts are not represented in unequivocal abstract or logico-mathematical terms.

It now is clear the child can no longer be considered a passive recipient of stimulation, nor can external reinforcement be considered a primary factor in learning and behavioral change. The introduction of "mediation responses" (verbal or otherwise) is not able to account for the complexities of observed changes in behavioral organization during the course of psychological growth during childhood. Many psychological theorists have adopted, in one form or another, the idea that human organisms actively respond to their environment and that the patterning of these responses reflects a "plan" or "set of cognitive operations." In other words, the child interprets environmental event input, but these interpretations are controlled by his capabilities for generating rule systems for coordinating and transforming the input to "match" a scheme, plan, or a mental operational structure. Analysis of the "rule systems" characterizing cognitive development, thinking, and learning requires specifications of the properties of, and antecedent conditions for, acquisition and structuralization of environmental events (mental representation/figurative knowledge) and of mental actions (operative knowledge) involved in coordination and transformation of those representations. The study of the development of rule systems defined as such is coincident, then, with the systematic investigation of the "inherent logic" of development of operational and figurative thought processes.

Intelligence, first of all, is considered no more, and no less, than biological adaptation; i.e., adaptation at any level of complexity reflects "intelligent" activity. "Knowledge" consists of two types of functional structures (figurative and operational) that construct invariants in organism-environment relations. These invariants are derived from abstractions from objects (physical experience) in the first case, and from coordinated actions (logico-mathematical experiences) in the second. Intelligence, then, refers to both types of cognitive learning and development and is defined in terms of functions (thus, thinking, reasoning) rather than content (i.e., words, verbal responses,, associations, etc.). Analysis of conditions for cognitive learning and development must begin with the identification of components of behavioral organization (structure) that reflect particular coordinated action-modes of the child as he is confronted with changing intrinsic (maturation and prior cognitive acquisitions) and extrinsic (physical and sociolinguistic) factors.

Cognitive structures or systems of coordinated (mental) actions proceed through invariant stages of structural change with autogenetic development. The successive differentiation and hierarchical integration of these cognitive structures permit the individual to cope with increasingly complex social and physical "realities." The process of cognitive development involves the changing characteristics of transformational rule systems (virtual and/or cognitive operations) characterizing the child's mode of adaptation. Neither the maturational structure of the organism nor the "teaching" structure of the environment is the sole source of reorganization; rather, it is the structure of the interaction between the child and the environment that provides the basic intellectual development; i.e., construction by the child.

Optimal conditions for structural organization and reorganization require: a) an optimal degree of discrepancy between environmental-demand structures and the functional-psychological structures -- both figurative (i.e., perceptual activity, images, memories) and operative; and b) social-learning conditions that demand "spontaneous" or "constructive" activity by the child.

Several implications for the construction of theoretically appropriate learning environments are implied in these general principles. First, structural change, for example, depends upon experience but not in a way that traditional learning theorists conceive experience; i.e., learning interpreted as pairing of specific objects and responses, direct instructions, modeling, etc. Rather, the functional genetic view holds that the cognitive capacities determine the effectiveness of training. For example, ability to solve class inclusion problems implies that the child already has the requisite single and multiple classification operational system for classes (i.e., combination, reversibility, etc.) in addition to appropriate information selection, storage, and retrieval abilities. At the same time, while experience is necessary for developmental progress, and appropriate enrichment of the environment can accelerate such development, experience cannot change the sequence, structuring, or emergence of action modes in the process of developmental change. In other words, organization of experience is not provided solely by the environment nor by the internal structures of the child.

Second, the structure of learning environment also must be considered relative to two frames of reference. The cognitive development of the child first must be analyzed in terms of the operational systems controlling his interpretation of environmental events. Such operations, or "transformation responses," are expressed behaviorally in the coordinated actions of the child as he is confronted with changes in the physical and social world. For example, the mental operations of associativity, reversibility, etc., can be inferred from the manner in which the child attempts to solve problems involving regular environmental contingencies (causality), understanding of spatial relations, etc. The content of substantive areas (e.g., science, mathematics) then must be analyzed and structured in terms of their own logical sequence and interlockingness with other contents. Certain concepts in the physical sciences, language, and mathematics, for example, each have their own inherent sequence and structure. Thus, certain concepts and information are necessary precursors to subsequent understanding of higher order concepts. Further, the interlocking nature of these "contents," in some cases, may be independent of the psychological state of the child. Optimal educational conditions require, then, thorough understanding of the psychological-cognitive capacities of the child as well as the sequential structuring of concepts within a particular curriculum area.

Third, the striving for equilibration between assimilatory and accommodative processes under both intrinsic or extrinsic pressures underlies the adaptive process. Optimum conditions for structural reorganization (learning in the broad sense) require disequilibration. This condition is met when there is an appropriate "mismatch" between the cognitive capacities of the child and the conceptual demand level of the learning task. Too little or too much "pressure" results in over-assimilation or over-accommodation tendencies respectively, but does not promote developmental changes in cognitive structures.

Fourth, facilitation of learning requires analysis of two levels of cognitive functioning -- figurative and operative processes. The first (i.e., figurative thought) is most emphasized by those theorists (particularly behaviorists) recommending a direct tuition approach to instruction. The operational theory of intellectual development does not deny the value of "provoked" learning (i.e., through imitation, algorithms, etc.). Rather, it must be recognized that such learnings are considered limited because of lack of generalization or transfer to new situations and because the basic (i.e., operational) intellectual processes concerned with problem solving and reasoning are not much affected.

While there is some doubt that much acceleration of structural reorganization is possible through environmental enrichment, early childhood education should provide opportunities for utilization of relevant cognitive operational structures. Generalization of conceptual learning across content areas rather than the building of specific knowledge and skills (e.g., a large vocabulary)should be emphasized since the latter cannot directly accelerate operational system change and may, in fact, retard development of these "deeper" competence structures.

In any case, the nature and variety of the child's "exchanges" with the environment need to be considered in educational planning. The nature of the interaction refers to the relative emphasis on autogenesis (self-directed) as contrasted to exogenesis (environmental or teacher-directed) structure of the learning environment. The position of the functional genetic position can best be summarized in the old adage -- "You can lead a horse to water, but you can't make him drink - unless you feed him salt." Thus, the task of the teacher is to engineer an educational environment consisting of curriculum materials, social interactions, and directed activities that provide appropriate "salt" for each child. Sequentially structured curricula could be designated to provide an optimum degree of environmental structure and level of conceptual material to permit appropriate balance of assimilatory and accommodatory activity.

The amount of interaction (i.e., enrichment) refers to the variety of structured curriculum contents which are relevant to the child's physical, social, and symbolic experiences. The interlocking nature of substantive curriculum areas makes it possible to provide a variety of experiences relevant to acquisition of the cognitive "products" that provide representation of the environment (memories, vocabulary, etc.) and, at the same time, to facilitate the development of coordinated rule systems associated with cognitive operational development. For example, analysis of the visual environment (attention or observational skills) as well as cognitive operational structures (e.g., conservation of area) can be emphasized in science, social studies, mathematics, art, etc.

The engineering of an educaitonal or "learning environment" based on the preceding consideration necessarily involves some specification of:

1) the child's cognitive developmental level; 2) the physical structures, including curriculum materials; and 3) the social or inter-personal structures. The organization of these "elements" should be such that the equilibration, between different cognitive systems and/or between intrinsic functional structures and environmental structures, is achieved. Thus, sequentially structured sets of curriculum materials and of social interaction situations are necessary to provide the "pressure" necessary for learning (adaptation). A variety of specific learning environments needs to be available to maximize the probability of each child finding activities that attract or "trap" him into interacting with the physical (e.g., science) and social (e.g., social studies) environment at both the behavioral and symbolic levels (e.g., language and mathematics) in creative and spontaneous ways (e.g., art, role playing, music). Finally, the physical and social environments should be arranged so that considerable freedom of movement, within the structure of a variety of contents, is possible, i.e., "a modified open-structure classroom." A careful balance between relatively high and low structured learning situations and between group and individual learning activities should be maintained.

The Mathemagenic Activities Program, a model developed in the context of enriching the educational environments of economically deprived children[4], is based on three explicit principles derived from the considerations discussed above. Specifically, the MAP principles of change -- whether the target for change is the individual organism (child) or a complex social system (e.g., Local Education Authority) are based on the above assumptions concerning the role of experience in learning and development. First, the source of motivation to change is provided by a discrepancy (disequilibration) between different conceptual systems (ideas) and/or between previously acquired conceptual systems and environmental task demands. Thus, an appropriate mismatch (M) is necessary to generate exploratory activities and insure the individual has the prerequisite conceptual basis for learning higher order concepts.

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[4] Mathemagenic Activities Program: A Model for Early Childhood Education, prepared by C. D. Smock (A preliminary statement of the conceptual basis for the Mathemagenic Activities Program for the Follow Through Program appeared in Terminal Report, Research and Development Center for Early Educational Stimulation, University of Georgia, August, 1970).

Second, since coordinated actions (practical and mental) are the bases for the knowledge acquisition, the learning environment must be structured so that specific task demands include appropriate practical, perceptual, and mental activity (A).

Third, the learning environment must include provisions for personal, self-regulatory (P)constructions. Knowledge acquisition involves construction of invariants from properties of objects (physical experience) and from the child's actions on objects (logico-mathematical experiences). Optimal conditions for facilitating new "constructions" (concepts learning) involve a balance between tasks that are highly structured (in which the child merely "copies" or imitates the correct solution) and tasks that permit the child to generalize and discover new applications of his concepts. Practically, self-regulation implies a variety of task options available to the child; the number of options may well vary with the nature of the task, level of development, and many other factors. However, MAP proposes that options -- in terms of level of task difficulty, mode of learning, and choice of activity itself -- are necessary ingredients of developmental change, whether the target be a child, a teacher, or an educational system.

The implied educational model requires significant changes in the teachers' role definition and teaching strategies (as well as tactics). The need for sensitivity to the child's capabilities, and the structuring of learning situations that promote self-regulated, "constructive" knowledge acquisition, together with thorough acquaintance with available technological aids, require an "educational engineer" in the best sense of that term.

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Elementary and Middle School Children's Comprehension of Euclidean Transformations

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MATHEMAGENIC ACTIVITIES PROGRAM - FOLLOW THROUGH

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PREFACE

Mathematics educators have been most receptive to investigations of a variety of instructional procedures for facilitating children's learning of mathematics. Only recently, however, have the traditional instructional sequences and children's intellectual characteristics to learn the prescribed concepts been questioned. Piaget's theory and research on the development of logical thought in children has provided a rich source of hypotheses about mathematical thinking of children. Dr. Kidder is one of a group of researchers at the University of Georgia who are examining the implications of Piagetian theory for mathematics instruction. His study is particularly relevant to the Follow Through program in that considerable emphasis is placed by MAP on matching the instructional tasks to the cognitive developmental level of the child and on learning mathematical concepts by actively manipulating or transforming concrete materials and then checking outcomes of these actions. The study of the children who have gone through the first three grades of the MAP program, at the age of the children in Dr. Kidder's study, should indicate the extent to which the child's learning strategies are maintained in other classroom contexts.

C. D. Smock
January, 1974

ABSTRACT

F. Richard Kidder

Elementary and Middle School Children's Comprehension of Euclidean Transformations

This study investigated nine, eleven, and thirteen-year-old children's comprehension of Euclidean transformations. More specifically, we investigated the children's ability to form a mental image of a planar figure, to perform a mental operation (Euclidean transformation) on this representation, and then to construct the resultant in terms of its correct position in space.

The results indicated that: (1) subjects in all age groups had great difficulty in performing Euclidean transformations, compositions of transformations, and inverse transformations at the representational level; (b) the majority of subjects in all age groups failed to conserve length while performing the transformational tasks; and (c) no general pattern of errors could be identified. The findings strongly suggest that the ability to perform Euclidean transformations at the representational level derives from what Piaget terms the stage of formal-operational thought.

Elementary and Middle School Children's Comprehension of Euclidean Transformations[1]

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A recurring question of interest to curriculum writers and mathematics educators is the following: "What geometry should be taught in the elementary and middle schools, at what age, and in what context?" Substantive recommendations for the inclusion of transformational geometry in the elementary and middle school curriculum have been made and are being implemented. However, there is a paucity of research data on which to base such innovations. More specifically, little empirical evidence exists concerning the spatial capability of children with respect to Euclidean transformations. Hence, the present study was devised to investigate nine, eleven, and thirteen-year-old children's comprehension of Euclidean transformations; that is, to investigate their ability to form a mental image of a planar figure, perform a mental operation (Euclidean transformation) on this representation, and then to construct the resultant in correct position. In formulating the study, primary consideration was given to the structure of geometry and to a comparison of this structure with what is known about the cognitive structure of the elementary and middle school child. Klein's principle, with its emphasis on transformations, invariants under transformations, and groups of transformations, concisely reveals the structure of geometry and is descriptive, in part at least, of cognitive operational structures as defined by the developmental theory of Jean Piaget.

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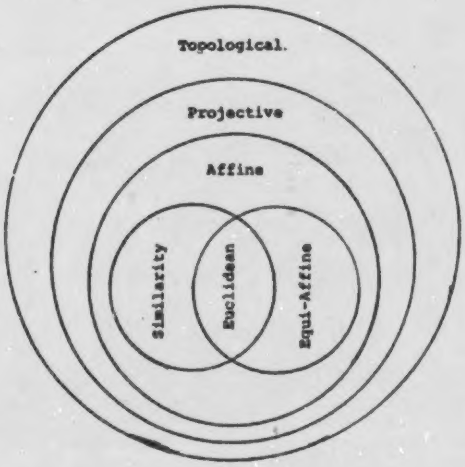
[1] This report represents a portion of the author's doctoral dissertation, conducted under the direction of Dr. Leslie P. Steffe, University of Georgia.

Mathematical Background

In 1872, Felix Klein synthesized the vast geometrical knowledge of his day with Arthur Caley's algebraic invariants and Sophus Lie's transformational groups into the now famous Erlangen Program, which not only defines geometries, but also classifies them as to content. A modern restatement of Klein's principle is: "A geometry is the study of the properties of a set S which remain invariant when elements of S are subjected to the transformations of some transformational group." Dorwart (1966) observed that since the advent of "abstract spaces" there are some very general systems which might be classified as geometries but which do not fit into Klein's principle. With this exception, Klein's principle leads one to a classification of geometries illustrated by the nesting of the defining transformational groups in the Venn diagram of Figure 1. An understanding of the role of the transformational groups in Klein's principle is enhanced by an

Figure 1

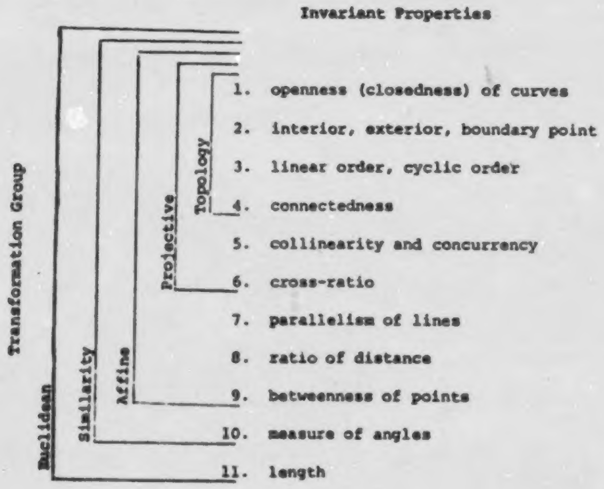
Venn Diagram: Nesting of Transformational Groups



examination of the transformations and some of the basic properties which remain invariant under such transformations. Figure 2 lists the most important properties being held invariant by the transformational groups. These eleven properties are not intended to be exhaustive but do, in a very real sense, exemplify the major differences in the geometries.

Figure 2

Properties Remaining Invariant Under Transformations


Invariant Properties
Transformation Group
Euclidean
Similarity
Affine
Projective
Topology
1. openness (closedness) of curves
2. interior, exterior, boundary point
3. linear order, cyclic order
4. connectedness
5. collinearity and concurrency
6. cross-ratio
7. parallelism of lines
8. ratio of distance
9. betweenness of points
10. measure of angles
11. length

Psychological Background

Three major themes are evident in Piaget's research on the child's development of space. First, Piaget's major focus in space development is space representation, not space perception. Second, he believes that these spatial representations are built up through the organization of mental "actions" performed on objects in space. And third, he claims that the child's earliest spatial concepts are topological in nature and that his projective and Euclidean concepts are concomitant extensions of these topological concepts. The first two, representational space and the organization of mental actions, are particularly relevant to the present study.

Laurendeau and Pinard (1970, pp. 8-17) give credit to Piaget for introducing the notion that a fundamental distinction exists between the various levels of knowledge implicit in the individual's contacts with the spatial world surrounding him. There are, according to Piaget, distinctions between perceptual, sensorimotor and representational space. Representational space is space in which the child can imagine or mentally reconstruct an action, and where the schema of these actions has sufficient mobility and flexibility for the internalized actions to become fully reversible mental operations. Piaget and Inhelder (1967) explain the distinction thus:

> "The evolution of spatial relations proceeds at two different levels. It is a process which takes place at the perceptual level and at the level of thought or imagination. ...Perception is the knowledge of objects resulting from direct contact with them. As against this, representation or imagination involves the evocation of objects in their absence or, when it runs parallel to perception, in their presence" (pp. 3, 17).

Firmly rooted in Piagetian theory are the words "actions" and "operations." Piagetians argue that cognition at all ages is best characterized as the application of real actions by the child. In infancy, these actions are externalized and observable for the most part, e.g., bringing thumb to mouth. As the child grows, the actions become more and more internalized, schematic, mobile, and less dependent upon concrete qualities. Gradually the cognitive actions coalesce to form systems of actions with strong structures. When they achieve this status they are called "cognitive operations" by Piaget (Flavell, 1963, pp. 164-173).

During middle childhood (7-11 years of age) there is an integration of the child's cognitive operations into system totalities with certain definite properties--reversibility, composition, and associativity--whcih suggested to Piaget corresponding logico-mathematical structures. Hence, he posited that a hybrid of the mathematical group and lattice structure was a good model for the structure formed by the concrete-operational child's system of cognitive operations.

As the child grows into adolescence, his concrete operational thought structures are extended by synthesizing the two forms of reversibility, inversion and reciprocity, into a single system of transformations (Inhelder and Piaget, 1958, p. 273). Flavell (1963) cites that now the logico-mathematical structures which serve as abstract models for the child's thought processes "...consist of integrated lattice-group structures, not just partial and incomplete lattice and group properties, as in the concrete operational groupings, but a full and complete lattice and a full and complete group, both integrated within one total system (pp. 211-212)." Piaget claims "...that the adolescent's behavior in certain problem solving situations attests to a cognitive structure with the properties of a four-group... (Flavell, 1963, p. 215)," consisting of the operation of I, the identity; N, the negation; R, the reciprocal; and C, the correlative (i.e., INRC group). Inhelder and Piaget (1958, p. 321) summarize their discussion of the adolescent's thought processes while solving mechanical equilibrium problems by noting that the INRC group comes into play in both its logical and physical form; i.e., the logical INRC form governs the propositional operations which the subject uses to describe and explain reality while the physical INRC form is revealed in the manner by which thought processes must be structured in order to solve problems.

The preceding theoretical discussion suggests the projective and Euclidean spatial framework of middle school children has developed sufficiently to enable them to perform Euclidean transformations, and that the mathematical four-group serves as a good model of the adolescent's thought processes. Hence, it seemed reasonable to assume that the group properties of the child's cognitive processes would be manifested in the context of Euclidean transformations; i.e., mathematical content with a similar structure. Therefore, the study was initiated to investigate the child's ability to: (a) form mental representations of planar figures, perform mental operations on these representations, and then construct the image of the transformation in proper position (performing Euclidean transformations at the representational level); (b) perform two such processes in succession (compositions of transformations); and (c) perform the process in reverse (inverse transformations).

Method

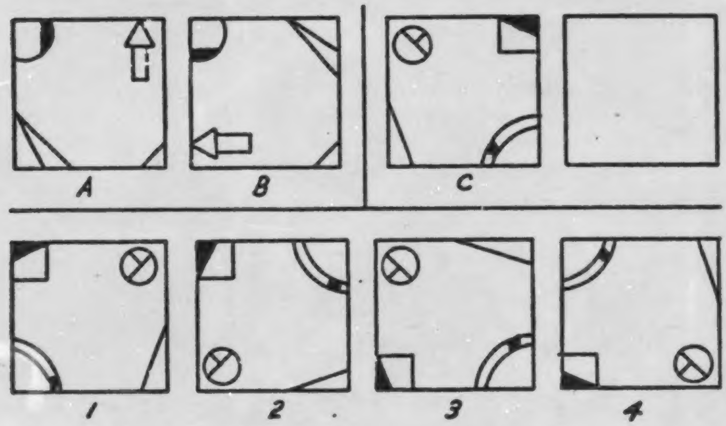
Ninety subjects--15 each from grades four, six, and eight in the Commerce, Georgia and Winder, Georgia, School Systems--participated in the study. Mean ages by grade as of November 1, 1972 were 9.3., 11.3, and 13.3 respectively. Students identified by teachers as being in the lower one-fourth general ability level were excluded. The Otis-Lennon Mental Ability Test Form J was administered to all subjects (after the experiment) in April, 1973. The mean raw scores for grades four, six, and eight were 43.6, 52.9, and 46.1 respectively.

Pretest

To determine subjects' ability to perform transformations at the perceptual-recognition level, a 20-item paper and pencil pretest was administered in groups by grade. The reptest was patterned after Alonzo's (1970) Spatial Analogies Problems Test in which she assumed, "...that two cognitive capabilities are needed to solve the test items: (1) the ability to identify the spatial transformation, and (2) the ability to reconstruct and select an identical transformation from the alternatives (p. 17." A prototypical pretest item is given in Figure 3.

Figure 3

Prototypical Pretest Item



This is a test of your ability to visualize designs as you change then around in your head. In each problem, first discover the relationship between square A and square B. Ask yourself, "What do I do to square A to make it look like square B?" Now imagine what square C will look like if you do the same thing to it. Select from the four possible choices, the square which is related to C in the same way that B is related to A.

Operational Definition Activities

Each subject was individually given an "operational" definition of translations (slides), reflections (flips), and rotations (turns). On the average, these activities required 10-15 minutes per subject. Using rigid figures and wire models of the motion indicators--slide-arrow, flip-line, and turn-arrow--the three motions were operationally defined thus: (a) an original figure and a motion indicator were placed in front of the subject, (b) a congruent copy was placed on top of the original figure, and (c) the motion was performed using the copy, leaving the original figure fixed. After a motion was demonstrated several times with various orientations, the subject was asked to perform the transformation as indicated by the motion indicator. A subject was considered operational if he performed three consecutive slides (flips, turns) correctly. If he made a mistake, his errors were discussed and he started over. After a total of five incorrect slides (flips, turns), he was considered non-operational on that particular motion. If a subject was found to be non-operational on all three motions, he was not tested further and was considered as having failed the entire transformational test.

Transformational Test

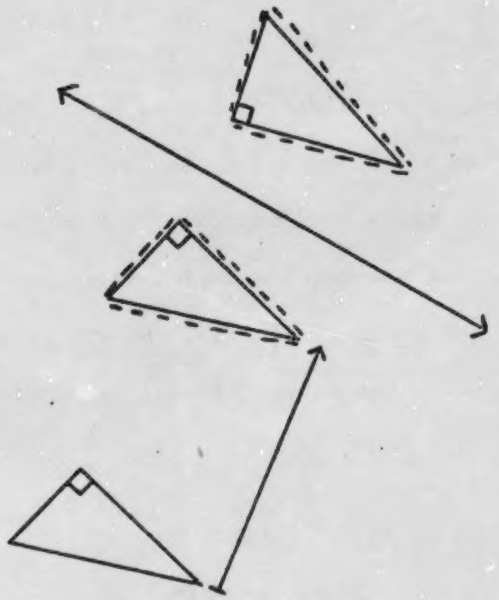
Immediately following the "operational definition" activities, those subjects found to be operational (72 of the 90 participants) were given the transformational test. The transformational test examined subjects' performance on individual motions, compositions of motions, and inverse motions at the representational level. The test was composed of tasks patterned after the protocols used by Piaget and his collaborators in their investigation of the child's development of space (Piaget and Inhelder, 1967). The three basic Euclidean transformations are slides (S), flips (F), and turns (T). Combinatorily these can be composed two at a time in nine ways (1) S-F, (1a) F-S, (2) S-T, (2a) T-S, (3) F-T, (3a) T-F, (4) S-S, (5) F-F, and (6) T-T. Using these compositions, the ordering of the tasks in the transformational test was as follows: For each subject, composition tasks

1 or 1a, 2 or 2a, and 3 or 3a were randomly selected and randomly ordered. Composition tasks 4, 5, and 6 were randomly ordered next, followed by the random ordering of the inverse tasks on slides, flips, and turns.

In a composition task, a subject was given an original figure and two indicated motions as exemplified in Figure 4.* Ten 1/8" diameter sticks, six for construction of two congruent images of the original figure and four of differing lengths as distractors, were placed adjacent to the compositional task. To insure that each subject was aware that he could construct an intermediate image, leave the intermediate image fixed, then

Figure 4

Prototypical Composition Task: Slide-Flip



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*In Figure 4, the intermediate and final images are dashed, indicating that these are not given--that they must be constructed by the subject.

construct the final image, and that he could measure (compare sticks with sides of original figure) he was told, "There are enough sticks to make two copies of the original triangle and you can measure if you want to." He was then asked to use the sticks and construct a copy of the original figure as it would look to him after performing the composition.

Regardless of success/failure on each composition, to examine capability with individual motions, each subject was asked to separately and individually construct the images of the two motions comprising the composition. After making the image of the first motion, the subject's construction was removed and a rigid congruent copy of the original figure was placed in correct intermediate position. The subject was then asked to construct the image of the second motion using the intermediate copy as the original figure of the second motion. To test the subject's capability with inverse motions, he was given a motion indicator and the image under the motion (but not the original figure). He was told, "This is how the figure looks to you after you have performed the indicated slide (flip, turn)."

Results

An Age X Motion repeated measurements analysis of variance was used to determine if Age and/or Motion were significant factors in the subject's ability to perform Euclidean transformations (Greenhouse and Geisser, 1959).

The ANOVA (Table 1) reveals that Motion was a significant factor ($p < .01$) in all modes of testing. Surprisingly, only in the pretest and the individual motions test was Age found to be significant ($p < .05$). Duncan's New Multiple Range Test was used as a post hoc procedure to isolate the above differences. Slides were found to be easier to perform than either flips or turns, but no significant difference was indicated between flips and turns. No significant pairwise contrasts were found in Age. Clearly,

Table 1

Summary of Analysis of Variance: Five Tests Comprising the Experimental Tasks

	SV	DF	F	
Pretest	Age	2	7.2968	$p < .01$
	Motion	2	11.1753	$p < .005$
	A X M	4	0.5855	
Operational Definition	Age	2	3.6569	
	Motion	2	28.3902	$p < .001$
	A X M	4	1.2666	
Individual Motion	Age	2	4.1723	$p < .05$
	Motion	2	22.3391	$p < .001$
	A X M	4	1.1634	
Composition	Age	2	0.9667	
	Composition	5	8.5368	$p < .01$
	A X C	10	0.2477	
Inverse Motion	Age	2	3.8737	
	Motion	2	10.8685	$p < .005$
	A X M	4	2.2432	

(see Table 3), the results did not indicate that the lack of age significance was due to the three groups already having the spatial capability to perform transformations at the representational level: the low scores on the transformational tasks indicated just the opposite. The only logical explanation is that the lack of age significance was due to the fact that few children of the three age groups were able to satisfactorily perform transformations at the representational level.

Pearson's product moment correlation coefficients were computed to examine relationships between the spatial ability levels of performing transformations. The pretest scores (perceptual ability) were not highly related with performance on the transformational tasks (representational ability). However, the data summarized in Table 2 indicates that a marked relatedness did exist between the scores on the Otis-Lennon Mental Ability Test and

Table 2

Upper Quartile (Q_1) Versus Lower Quartile (Q_2)

Performance on Experimental Tasks[a]

Age[b]		Otis-Lennon Mean Scores	No. individual motions performed[c]	No. composition performed	No. inverse motions performed[e]
9	Q_1	68	20	7	3
	Q_4	25.1	5	1	1
11	Q_1	68.3	35	7	9
	Q_4	36.3	5	2	2
13	Q_1	63.9	37	8	11
	Q_4	28.6	1	0	0
Total	Q_1		92	22	23
	Q_4		11	3	3

a. as determined by score on Otis-Lennon Mental Ability Test.

b. seven subjects in each quartile per age.

c. twelve individual motions possible for each subject.

d. six compositions possible for each subject.

e. three inverse motions possible for each subject.

the ability to perform transformations at the representational level. A comparison of the thirteen-year-old lower quartile subjects' performance with that of the nine-year-old upper quartile subjects' performance on the transformational tasks is striking. The thirteen-year-old lower quartile subjects were only able to satisfactorily perform one individual motion out of a possible 84 and no compositions or inverse motions. In contrast, the nine-year-old upper quartile subjects performed 20 individual motions, 7 compositions, and 3 inverse motions. The contrast in relatedness of pretest scores and Otis-Lennon Test scores with performance of the transformational tasks indicates that the pretest measured specific spatial abilities whereas the Otis-Lennon Test measured more general reasoning abilities. Furthermore, it is suggestive that, in addition to spatial ability, broad general reasoning ability is needed to perform Euclidean transformations at the representational level. These findings are, in general, confirmatory of Alonzo's analysis of the relation of spatial to "logical reasoning" abilities (1970).

A descriptive analysis was used to assess subjects' ability to perform transformations. It was found that subjects had difficulty in performing transformations in all modes of testing.

In particular, as shown in Table 3, the majority of subjects in each age group failed to perform Euclidean transformations, compositions of transformations, and inverse transformations at the representational level.

Table 3

Performance on Transformational Test by Age Group

Number and Percent of Motions Performed*

Age	Slides		Flips		Turns	
9	24	20.0%	11	9.2%	8	6.7%
11	34	28.3%	20	16.7%	19	15.8%
13	53	44.2%	30	25.0%	23	19.2%
Total	111	30.8%	61	16.9%	50	13.9%

*There were 120 slides, flips, and/or turns possible for each age group.

Number of Ss Satisfactorily Constructing Composition Images

Age	S-F	S-T	F-T	S-S	F-F	T-T
9	1	1	0	6	1	1
11	3	1	0	6	1	1
13	4	1	2	8	2	2
Total	8	3	2	20	4	4

Number of Ss Satisfactorily Constructing a Given Inverse Motion Image

Age	Slides		Flips		Turns		All Three	
9	4	13.3%	1	3.3%	2	6.7%	1	3.3%
11	8	26.7%	4	13.3%	5	16.7%	3	10.0%
13	14	46.7%	9	30.0%	5	16.7%	4	13.3%
Total	26	28.9%	14	15.6%	12	13.3%	8	8.9%

The inability of subjects in all age groups to perform individual motions (the basic elements of the Euclidean transformational group) precluded an examination of the experimental question, "Will the four-group, which Piaget hypothesizes serves as a model for the cognitive structure of the adolescent, manifest itself when the adolescent is confronted with mathematical content which is similarly structured?"

An error analysis was undertaken to determine the kinds of errors committed during the transformational tasks and to identify possible error trends by individuals. No error pattern could be established, either on motions or by individuals. However, it was found that 70% of individual motion errors, 67% of composition errors, and 59% of inverse motion errors were "failures to conserve length" (did not construct a congruent image of the original figure). When the percentages of length errors were averaged across motions, it was found that 69%, 71% and 70% of the 9-, 11- and 13-year-old subjects respectively, committed length errors on the individual motions tasks. Strikingly, only four of the 72 subjects attempting the transformational tasks were able to perform all 12 individual motions with no conservation of length errors.

Discussion

The data did not support the experimental hypotheses that adolescents could perform Euclidean transformations, compositions of transformations, and inverse transformations at the representational level. The findings are not consistent with the theoretical base for the study nor with results of previous studies. Piaget (1967) contends that by the age of nine or thereabout the child has the framework appropriate for comprehensive Euclidean and projective systems. Replications of Piaget's spatial tasks by Dodwell (1969) and Lovell (1959) note some variation in age but, in general, support Piaget's position. Laurendeau and Pinard's (1970) findings are also supportive of Piagetian theory; however, they observed the emergence of the coordination of perspective ability to occur slightly later.

Little research on the child's ability to perform Euclidean transformations as such is available. St. Clair (1968) found that fourth and sixth grade students can learn the concepts of symmetry (reflections and rotations). In a training study using students from grades four and five, Turner (1967) reached the broad general conclusion that the teaching of symmetry and related principles has potential for increasing children's spatial capability. And, Williford (1970) found that second and third grade children could be taught transformational concepts but that there was little transfer to other spatial abilities.

The above comparison forms a basis for the experimentor's conjecture that the ability to perform transformations at the representational level derives from formal-operational thought (in a Piagetian sense) and that the 13-year-old subjects of the present study were not in the formal operational stage. This conjecture is supported in part by the preponderance of "failure to conserve length" errors committed by all age groups during the transformational tasks.

Piaget's experimental technique in examining the child's conservation of length (classical definition) consisted of showing the child two sticks of the same length lying side by side, asking him if they were equal in length, sliding one of the sticks forward approximately 1 cm., and again asking the child to say which of the two sticks was longer or if they were still the same length (Piaget, Inhelder, and Szeminska, 1960, p. 95). Piaget does not give the ages of the children he found in each stage; however, the protocols indicate that six to eight years of age was the range for emergence of consistent conservation of length. Numerous studies also imply that young children, ages 6-8, do in fact conserve length. Shantz and Smock (1966), Divers (1970), and Carey and Steffe's (1972) findings infer that young children conserve length in the classical sense.

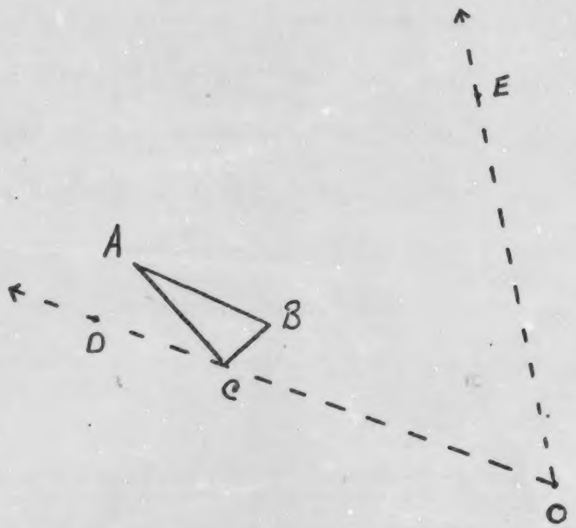
Why, then, was this capability not evidenced during the present study? Piaget and others established a child's capability to conserve length with two sticks as described above. The studies referenced inferred length conservation from the acquisition of some other conservation capability such as transivity or seriation. In contrast, the present study used three types of transformations, six different compositions, and three types of inverse transformations. And, instead of simple short sticks, length had to be conserved with a complex point-set planar figure. That is, the subject had to hold invariant the shape and size of different prescribed three-sided figures under much more complicated transformations than just a 1 cm. slide of one stick. In the "classical" test, a simple comparison could be made between the stick which was moved and the stick which was not moved. No such comparison of corresponding sides of the original and image figures could be made in the present study. A subject had to be aware of the necessity to hold length invariant at the outset, and he had to select the proper length stick with which to construct the image figure prior to making the transformations.

In addition to the large number of "failure to conserve length" errors, the almost complete lack of error pattern also suggests that the ability to perform transformations at the representational level derives from formal-operational thought. One general property of formal thought is the ability to separate factors not given by direct observation and the necessary tools for holding one factor constant in order to determine causal action of another (Flavell, 1963, pp. 204-211). It appears that just such a capability is needed to perform Euclidean transformations at the representational level. The many factors involved, the noting of factors not given by direct observation, and the holding constant of some factors while varying others is best seen by example.

To rotate a triangle as depicted in Figure 5 and to satisfactorily construct the turn-image, a subject must (at least): (a) be aware that length of sides AB, AC, and BC remain invariant under the turn; (b) be able to imagine the existence of rays OD and OE, and the existence of angle ACD; (c) be aware that angle ACD is held invariant under the turn; (d) be able to rotate the figure through angle DOE; and (e) hold distances OC, OB, and OA invariant while rotating the triangle. In Figure 5, rays OD and OE are dashed, indicating that these are not given--that they must be imagined by the subject. If the subjects were at the concrete-operational stage of reasoning (or even in a transitional stage) it is to be expected that they would focus attention, first on one variable and then on another in performing the transformational tasks. This would, of course, explain why length was sometimes conserved and sometimes not, and why errors on previous tasks of the same motion were not necessarily repeated on a current task.

Figure 5

Prototypical Rotation



58

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On a Model For Learning and Teaching Mathematics

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of the

University of Georgia

Charles D. Smock, Director

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Preface

The Mathemagenic Activities Program is a model for early childhood education, sponsored by the National Follow Through Program, based primarily on Piaget's theory of cognitive development. The emphasis on structure of cognitive operational systems (logico-mathematical experiences) as the basis for understanding intellectual development provides mathematics educators with a unique opportunity for potential contributions to child development. This paper, also appearing as the first chapter of Research on Mathematical Thinking of Young Children (Six Empirical Studies), explores some of the implications of Piaget's theory for mathematics learning and teaching.

C. D. Smock, Director

On a Model for Learning and Teaching Mathematics

by

Leslie P. Steffe

and

Charles D. Smock

Piaget (1971, p. 21; 1973a, p. ix) has expressed the convictions that (1) experimental pedagogy must remain distinct from psychology but, yet, utilize psychological principles, and (2) hypotheses derived from psychology must be subjected to empirical verification (or refutation) rather than be accepted only on the basis of deduction. While it would seem unnecessary to restate these two convictions, education has a history of embracing a particular theory or point of view based on overly simplistic deductions and then, when educational practice has shown the transparency of these deductions, abandoning that theory. Piagetian cognitive-development theory may be no exception to this general pattern in education because it is, in the main, the theoretical basis on which the "mathematical laboratory" is built.

The mathematical laboratory, according to Smock (1973) has "remained only loosely or ambiguously defined (p. 1)." If that assertion remains true after substantial attempts are made at an unambiguous definition, then the mathematical laboratory would have to rely on only sloganese or personal testimony for its justification as an instructional approach. Until and unless mathematics educators return to the beginning and ask "how do children learn?" rather than "how do we teach?" the fundamental dimensions of educational and instructional problems facing the mathematics educator will remain unidentified. If we confront this basic question, it is possible to proceed with the task of characterizing what is meant by a "mathematical laboratory". The emphasis, then, is to be placed not on methodology and learning activities but, rather, on the learning characteristics of the child as he acquires particular subject matter and content and methodology constructed for that special purpose.

The problems of characterizing the mathematical laboratory can best be understood in the context of the issues involved in developing a theory of mathematical learning and instruction. Bruner points out (1964b) that most theories of learning and development are descriptive rather than prescriptive. While this may be the case, a theory of mathematical instruction for children must be based in the developmental constraints of concept learning, a theory of learning relevant to mathematical concepts, and in mathematics itself if the theory is to have validity. Consequently, rather than focusing on instruction with little regard for learning or on learning with little regard for instruction, analysis of school mathematics requires that learning and instruction be considered simultaneously. Even then, little progress will be possible if mathematics and cognitive development are ignored.

One of the best examples of an analysis and synthesis of cognitive development theory, fundamental mathematical structure, and mathematics instruction for the early school years has been presented by two Russian educational psychologists, El'konin and Davydov (1974). Following Vygotskii, a child's mental development is viewed as being ultimately determined by the content of the knowledge studied. They feel that researchers who study the development of mental operations (notably Piaget) concentrate only on those processes which are maximally independent of specific subject matter. El'konin and Davydov are critical of this approach because it leads to a view that the sources of mental development lie in the individual independently of the specific historical conditions of existence and characterize the child's mind in absolutist terms. On the other hand, they do not ascribe to Bruner's hypothesis that "any subject can be taught effectively in some intellectually honest form to any child at any stage of development." Such an assumption makes reference to abstract forms of teaching fundamentals of any subject to a child of any age, but forms of instruction must be found that are suitable for each specific piece of content and given age level.

Piaget (1971, p. 21), however, clearly differentiates experimental pedagogy from psychology. Experimental pedagogy is concerned less with the general and spontaneous characteristics of the child than with their modification through pedagogic processes. Moreover, in commenting on the value of developmental stages in educational sciences, Piaget (1971, p. 171) rejects the notion of inflexible stages characterized by invariant chronological age limits and fixed thought content. As an interactionist, Piaget (1971, pp. 171-73) advocates that the cognitive structural changes that come about through maturation and those that derive from the child's individual experience be considered as separate factors in intellectual development. Also, Piaget (1964a) maintains that mathematical structures can be learned if the structure of interest can be supported by simpler, more elementary structures. Consequently, while the Genevans' work has not been in experimental pedagogy but has dealt with the development of the child independent of particular subject matter curricula, experimental pedagogy does not stand in opposition to cognitive-developmental psychology. Rather, experimental pedagogy is complementary to it, with the potential of contributing knowledge of the developmental processes.

Piaget rejects the notion of inflexible stages characterized by invariable chronological age limits and fixed thought content. Yet, El'konin and Davydov make a central issue out of whether to characterize a given age level in terms of processes for which a developmental period is concluded or in terms of the processes for which that developmental period is beginning. The former is rejected because it would lead to presentation of educational exercises that demand only previously formed intellectual processes for solution and to the further assumption that intellectual development is inviolable and independent of the content and methods of presentation of subject matter. El'konin and Davydov, thus, believe that development of psychological processes underlying the learning of mathematics not only do not precede instruction in mathematics but are formed in the process of learning.

El'konin and Davydov's point of view is consistent with the leading role ascribed to instruction by Soviet psychologists (Kilpatrick and Wirszup 1969, p. v). But El'konin and Davydov do not separate methods of instruction from the content of what is to be learned nor from the general cognitive development of children. El'konin and Davydov test experimental curricula based on measurement processes, but their experiments do not provide conclusive evidence that the cognitive development of the children was altered. The experiments shed some light on symbolization capabilities of concrete operational children in highly structured measurement exercises, but little evidence is presented which would lead one to believe these concrete operational children were, as a result of the instruction in measurement, in the formal reasoning stage vis-a-vis Piaget. Thus, the contribution of school learning to cognitive development (in a Piagetian sense) remains to be adequately tested.

The important contribution of El'konin and Davydov is the detailed analysis of fundamental mathematical structures and the explication of the yet unconfirmed hypothesis that cognitive development of children can be altered by school instruction in mathematics. Their analysis of the fundamental mathematical structures led them to the conclusion that the concept of quantity, with its roots in the ordering structures, should be the starting place for school mathematics. The task of analyzing the similarities and differences of the fundamental mathematical (including those mentioned by El'konin and Davydov) and genetic structures is an important first step (Beth and Piaget, 1966).

One important difference is that mathematical structures are the object of reflection by the mathematician but genetic structures are manifested only by the child's behavioral-action structures, which are determined by his assimilation of past experiences. A second major difference is that form is independent of the content in the mathematical structure but in genetic structures the form is inseparable from content. Finally, in the mathematical structures axioms are the starting point of formal deduction whereas in genetic structures the laws are the rules which the child's deductions obey.

Similarities in the structural types are twofold: (1) operations in the mathematical structures correspond to operations in the genetic structures; and (2) the axioms of the mathematical structures correspond to the "laws of combination" in genetic structures. It is these operational genetic structures which El'konin and Davydov identified as being maximally independent of specific subject matter. The basic content of the genetic structures, however, are classes and relations which, in synthesis, form the basis of cardinal and ordinal number in the logical domain and of quantity and of measurement in the infralogical domain. On the face of it, then, it seems as if El'konin and Davydov's declaration of genetic structures as maximally independent of school mathematics is unjustified. It is true, however, that profound structural differences exist in the genetic structures of concern and certain of the mathematical structures which are used as the basis for the content of school curricula. These structural differences are a central issue in the determination of the applicability of genetic structures to mathematical learning. Mental operations associated with such mathematical structures may or may not be accounted for by the mental operations ascribed to the genetic structures. If the latter case is so, El'konin and Davydov may be essentially correct in asserting that the psychological processes are developed concomitant with learning. Piaget (Beth and Piaget, 1966, p. 189), however, proposes that the construction of mathematical entities is an elaboration of the elements of natural thought and the construction of mathematical structures is an enlargement of particular mathematical entities.

The Piagetian hypothesis is, at least, intriguing--it suggests that the individual constructs his own mathematics. The testing of the hypothesis, however, is extraordinarily complex due to the structural differences alluded to above. So, two programs of research would seem to be necessary. On the one hand, the logical and mathematical veracity of Piaget's system, concentrating on similarities and differences between genetic and mathematical structures, requires both logical and empirical study. On the other hand, models (albeit preliminary) for learning and instruction of mathematical concepts need to be developed where that development explicitly includes theoretical as well as empirical components. It is this second program of research which is of basic concern in this monograph.

Piagetian theory offers several principles which can be utilized in the construction of such a model (Piaget, 1971, 1973b; Smock, 1970, 1973). First, equilibration theory provides a theoretical model of knowledge acquisition with specification of the factors that regulate acquisition. Second, two distinct levels of cognitive functioning--figurative processes and operative processes--are identified as necessary for understanding learning and development. Third, cognitive capacities determine the effectiveness of training and these cognitive capacities are influenced by four factors that contribute to cognitive development. Fourth, the learning environment must be considered from both the points of view of the genetic structures and the mathematical structures. The implication of some of these principles for the development of an instructional model and for research relevant to that model is presented below.

General Factors Contributing to Cognitive Development

Piaget (1964a) has identified the major factors contributing to the development of cognitive growth of children as including: (1) maturation, (2) experience, (3) language, and (4) equilibration.

Maturation

The proposition that maturation is a major determinant of cognitive growth is not, of course, a novel idea. What is new in Piaget's analysis is the explicit inclusion of a maturational component as a modulating factor to the experimental and/or experiential contribution to development progress. The constraining role of maturation is supported by, for example, the fact that transitive reasoning seldom has been observed in children below four years of age. The evidence indicates great difficulty in a child's learning transitivity of, for example, "as many as" during the stage of preoperational representation (even when apparently appropriate learning experiences have been encountered). Such evidence cannot be taken as proof that maturation is responsible, but it certainly suggests that instrinsic physio-biochemical processes play a prominent part in development of thinking. Subjecting a child to learning "experience" does not appear sufficient to insure he will understand the concept of transitivity.

Experience

Experience in and of itself may not be sufficient to explain conceptual learning of children but no one denies its importance for intellectual growth. But, if experience was sufficient, all one would have to do to "teach" transitivity would be to give the child sufficient exposure--and he would learn. But, unfortunately, it is not that simple.

Physical Experience and Mathematical Experiences. Piaget (1964a) has analyzed "experience" into two components: physical and logico-mathematical experience. Imagine, for example, that a child matches objects of Set A one-to-one with objects of Set B through overt (practical) actions; i.e., he places one object from Set A in correspondence with one object from Set B, etc., until all the objects of one (or both) of the sets are exhausted. Then, he takes the objects of Set B and likewise matches them with the objects of another set, C. Does this matching constitute a physical experience or a logico-mathematical experience? It could be either, depending on the cognitive level of the child.

One cannot differentiate between the two types of experiences through observation of the child's overt acts of matching. The crucial determiner of the type of experience is whether the Sets A and C are "related" by the child by virtue of the comparisons of A and B and then, B and C. If the child is not able, through reasoning (mental operations), to determine the relation between A and C, then the experience gained through overt matching of the objects of A and B and B and C was, by definition, physical in nature. The relation between the Sets A and B in this case was a function of the physical arrangement of the objects and would not exist for that child in the absence of perceptual input. That is, relation remains external to the child and thus is destroyed upon rearrangement of the objects of the sets. When the two sets of objects are in a state for physical comparison, the child definitely obtains knowledge about the objects--either they match or they don't--but, for that knowledge to be mathematical in nature, the relation must be conserved by the child when the objects are moved to new states and the child must be able to engage in reasoning involving the properties of the relations that go beyond the perceptually or graphically "givens."

The distinction between a physical experience and a mathematical experience is essential to the understanding of the growth of mathematical concepts. Knowledge based on physical experience alone is knowledge of static states of affairs and, if a child is wrong, it is easy to demonstrate that to him. However, knowledge derivable from mathematical experience is another matter; if a child is wrong, it is difficult, if not impossible, to convincingly demonstrate, or even to get the child to accept verbal explanation of the correct answer. For example, if a child fails to align the two endpoints when comparing length of sticks, it is quite easy to correct the mistake. If, however, he fails to display transitive reasoning in a task, one or two examples is not likely to teach him the concept.

Physical knowledge, then, is the construction of the invariants relevant to the properties of objects (i.e., states) and is based on "experience" through direct contact with objects through one or more of the five senses. For example, one may touch something and it is hard, cold, hot, soft, supple, etc. Or, one may see something--an object is red, a diamond cutting glass, the shape of a banana, etc. The source of mathematical experience, however, is assumed to be the abstractions from coordination of actions vis-à-vis object; i.e., transformation of the "states" associated with series of discreet physical experiences. The critical difference is that the mathematical knowledge gained demands that a pair (or set) of physical objects not be defined by the temporal-spatial (perceptual) similarities, but rather by the invariant relations among or between objects. Overt (perceptual) actions alone are not sufficient for mathematical experience. As already noted, a child may match the objects of two equivalent collections and the knowledge gained from the actions and perceptual consequences may be no more than physical knowledge. Often cited as an example of mathematical experience is the realization by a child that it makes no difference how you count a collection of objects--you get the same number. Such knowledge is gained only through counting the collection in at least two or more ways; i.e., coordination of mental, as well as practical, actions.

Linguistic Transmission

Language, the third factor in the growth of mathematical concepts, is considered a part of the experience of the child, but deserves special consideration because of its special quality and status within the total realm of experience. Information contained in a verbal communication will not, necessarily, increase a child's understanding of a mathematical concept. Bailey (1973), for example, presented a transitivity problem to 40 first graders, 40 second graders, and 40 third graders who were in the top two-thirds of their class according to teachers' judgments. Each child was presented a transitivity of length problem. Those children who did not solve the problem were told the correct relations between the two sticks. For example, if a child took a stick B and first compared it with C, and then with A, and found that A and B were the same length and B and C were the same length, but could not infer the relation between A and C, he was told A and C were the same length. After the verbal instructions, the child was asked to explain why A was as long as C. Of 24 children who could not infer the correct relation between A and C, only five would give a satisfactory explanation of why A and C were of the same length after the verbal instructions.

It would appear that for children in the stage of preoperational representation (or even in the transitional stage), any attempt to teach mathematics concepts only through verbal or symbolic means will be unsuccessful. But, because words and symbols are an important part of mathematics their specific functions must be clarified. Until then, a carefully arranged interplay between the spoken words which symbolizes a mathematical concept and the sets of actions performed in the process of constructing a tangible representation of the concept should be maintained. In short, a mathematical vocabulary should be developed during the course of activities used to explicate and provide embodiments for a concept to be learned. The particular blend, of course, will be determined by the specific activity and characteristics of the child.

Equilibration and Learning

Of the four factors which contribute to the growth of mathematical concepts, Piaget considers equilibration to be most fundamental. Equilibration is a self-regulatory mechanism that balances the invariant biological adaptation processes of assimilation and accommodation. Assimilation refers to the process by which novel events are integrated into the existing mental structures. The complementary process of accommodation concerns the alteration of mental structure under the pressure of this new information.

Learning, in this context, refers to the process by which new information is assimilated into available cognitive structures and to the modification of those structures (accommodation). Mathematical learning, then, appears to be more than association of stimulus and response. The association of 6 with (2 x 3) is important and no one doubts it can be conditioned (or memorized) using appropriate instructional strategies. Teaching which assumes a stimulus-response view of learning runs the risk of promoting physical knowledge and not mathematical (or operational) knowledge. However, because learning is generally thought to be provoked by situations external to the learner, it is necessary to analyze the levels of learning relevant to particular mathematical concepts or structures.

If our assumption is correct, there should be a differential emphasis on assimilation and accommodative activity depending on the level of understanding of a particular concept. Assimilative task structures would emphasize "play" and/or self (child) regulated activities until the child's behavior indicates the essential elements of a concept have been assimilated. Then, more task situations (including modeling and verbal exploration), as well as situations designed to utilize and generalize the relevant concepts to new situations, should be introduced. The appropriate balancing and sequencing of the assimilative-accommodative activities (practical and mental) requires considerable theoretical and observational skills of the instructor. At the same time, the basic ideas and associated techniques can be identified and used as guidelines for a Piagetian type of learning environment.

Levels of Mathematical Concepts. Relations, classes and number take a relatively long period to develop as operational concepts in the child, appearing perhaps as late as nine to ten years of age. The stages of cognitive development in Piagetian theory are identified as sensori-motor, intuitive preoperational, concrete operational, and formal operational with the movement from one stage to the next being clearly marked by "transitional" phase characteristics. Our assumption is that mathematical concepts go through similar "stages" (called levels) as the child "learns" a concept; e.g., in the case of relations, either children have little or no knowledge of relations, or they are able to engage in reasoning involving the properties of the relations, or they are oscillators--at times, in restricted situations, they appear as if they are able to reason involving relations, but that reasoning is limited and can be extinguished quite easily. Finally, as the assimilation-accommodation activities are "balanced," the child now "feels" he "understands" and insists on the "logical necessity" of the concept.

Learning-Instructional Phases for Mathematical Concepts

Phase I: Exploration

Equilibration, the balancing of assimilatory and accommodatory activity, is a useful theoretical construct to help determine the criteria for learning activities in mathematical instruction. Exploratory activities have been identified by Piaget as representing a preponderance of assimilatory activity; i.e., modification of the environment to match the existing cognitive structures. As such, exploration is conceived as an essential first step in mathematical instruction and learning. This first phase corresponds to the first level of mathematical concepts identified; i.e., to essentially "no concept" and to the emergence of the second level (rudiments of a concept). At these two levels of mathematical concept development, emphasis is on the constructive thinking by the child. It is a period of concept formation and not analysis. Exploratory activities can vary along two dimensions: the type and structure of the material and the degree of direction given to the child. However, it is important to keep in mind that the child needs to structure (assimilate) the activities but in a direction relevant to the particular concept learning desired.

Multiple Embodiment Principle. In order to illustrate the above principles using particular concepts, imagine that one-to-one correspondence is the concept of interest. The "no concept" phase of one-to-one correspondence corresponds exactly to the preoperational stage of development. Children who display lack of one-to-one correspondence first must be allowed to engage in undirected exploratory activities, using physical objects that later will be used in more direct activities. For example, the child may be given assortments of beads, bird cutouts, blocks, discs, animal cutouts, toy animals, toy cowboys, toy soldiers, etc., and allowed free play time with these materials. Most children will attempt to place cowboys and Indians on horses, dress dolls, stack dishes, stack blocks, string beads, categorize animals or bird cutouts, align soldiers in rows and give each guns, etc. Further, this type of practical play is extended by the child into symbolic play; i.e., such as waging wars, keeping house, setting tables, etc. The length of time and number of free play activities which should be encouraged is determined by the frequency of "transitional indication" in the child's behavior and two additional principles: (1) multiple-embodiment and (2) mathematical variability (Dienes, 1971).

The first (multiple-embodiment) states simply that in play activities, the child should use as many different material sets as appropriate so long as each material set is conducive to construction (by the child) of the concept. Consider a particular free play activity where the preoperational child places cowboys and Indians on horses. Through this assimilatory activity, the child can gain the physical knowledge that indeed the cowboys and Indians fit on the horses. The situation is set for an adult to create a disequilibrium for the child by asking the child if there are enough cowboys and Indians so each horse would have a rider. To find out, the child has to change his practical activity to answer that question. Now if the child correctly achieves that task, the next step is to introduce new materials but maintain the original goal (one-to-one matching); e.g., are there enough dresses so one could be put on each doll?

If the child does not initiate specified goal-directed activities following suggestions, the adult may then demonstrate that there are enough cowboys and Indians so each horse has a rider. Such imitation learning (accommodatory activities) provides the conditions to employ the principle of multiple-

77

embodiment for subsequent learning tasks and even, at times, imitative behavior across task situations. The adult must, always, be sensitive to the type of knowledge the child is acquiring in the imitative activities; i.e., knowledge acquisition under imitative conditions has a high probability of being at the level of "physical knowledge" if children are in the pre-operational stage. By remaining sensitive to this, the adult will avoid expecting the child to build an understanding of higher-order concepts prior to acquiring the necessary prerequisite concepts.

Mathematical Variability Principle. It was pointed out above that free play activities can, through appropriate intervention, be changed into more directed learning activity for children. The second dimension of task activity that can be modified by the adult involves the application of the mathematical variability principle (Dienes, 1971). In contrast to the multiple embodiment principle where the mathematical content is held constant and the materials varied, the mathematical variability principle varies the mathematical content. For example, in case of one-to-one correspondence, the relation being considered can either be changed to a new relational category altogether (e.g., length relations or family relations) or to a relation within the category of matching relations (i.e., more than, fewer than, as many as). Both of these types of variations can, if appropriately used, create cognitive conflict to be resolved by the child.

In summary, it has been pointed out that "play" can vary from free play to directed play, where the directed play is a natural extension of free play. Further, the principles of multiple embodiment and mathematical variability provide guidelines for setting the stage for transition from the level of physical experience to mathematical experience. However, dramatic short-term success in teaching mathematical concepts to preoperational children is not to be expected. More success can be expected with children in the transitional stages (which corresponds here to the second level of mathematical concepts), but again, the short-term success will undoubtedly be modest. During the developmental phase of preoperational representation, it is advocated that the *learning-instructional* phase conditions be held relatively constant; i.e., use the explorations with variations based on the mathematical variability and multiple embodiment principles.

Phase II: Abstraction and Representation

The second learning-instructional phase, that of abstraction and representation, is based in part on the distinction between physical experience and logico-mathematical experience. Abstraction from properties of objects (physical experience) are called simple abstractions; e.g., hardness, sharpness, etc. Reflective abstraction (logico-mathematical experience) involves abstraction from the *actions performed on* (or with) objects *or* representations of those actions. Reflective abstraction is represented by the case of a child counting a string of beads from one end, then from the other, and realizing that the number of beads is independent of the manner of counting them; i.e., which is selected first, which second, which third, etc. The beads are there, but the knowledge gained had to do with the actions with the beads and the capability of representing and viewing the actions. Or a child may pair elements from two sets until one is exhausted prior to the other; re-pairing the elements in a different way provides the conditions necessary to make the abstraction that no matter how the pairing is done, the one set will always contain more elements than the second set. It is clear that a child may be "playing" but still be engaged in reflective abstraction. Thus, the teacher need not be restricted to "play" activities as long as the child reveals capacity to make the higher level abstraction.

The child may engage in reflective abstraction but not achieve stable representation of newly gained knowledge. Representations may be figurative (e.g., images) derived from drawings or perception of a collection of symbols, etc., and not available to the child at a later time. For example, if a child compares a green stick with a red stick and finds the green stick to be shorter than the red stick, any one of

G < R, G < R, or G R

might be used as a static, immediate representation. There, reflective abstraction and representation together contribute to a higher level of concept formation than expected in the first learning-instructional phase in that rudiments of mathematical concepts are present and can be utilized in a limited way (2nd level of concept formation).

Phase III: Formalization and Interpretation

The learning-instructional phase of formalization and interpretation completes the proposed learning cycle for mathematical concepts. The mathematical concept base-ten numeration system will be used to illustrate the three learning-instructional phases, with emphases on formalization and interpretation (Phase III). The concept was selected because no data are presently available that show the numeration systems are part of the natural cognitive development of children.

We assume the child is at the concrete stage of operations and the operational structures related to classification and relations are available to him. It must be noted that because a child is at this developmental stage does not mean he knows base ten numeration nor that figurative representation of the concept has been achieved. What use the child is able to make with his knowledge of classes, relations, and number may be in the absence of formalization of numeration concepts. Therefore, the only prerequisites required are the completion of a learning cycle concerning the ordering of the digits 0, 1, 2, 3, 4, 5, 6, 7, 8, and 9; ability to write them from memory, and to do simple addition problems.

Any natural number can be written in expanded notation. Consequently, the coefficients, the base, and the exponents can all be allowed to vary in employing the mathematical variability principle. Usually, the base is held constant (base ten is used) and only the exponents and coefficients are allowed to vary.

Imagine that a small group of children are given a collection of various assortments of materials, such as geometrical shapes, checkers, dried beans, etc., and are allowed to engage in free play with the materials, building whatever they wish--castles, houses, roads, forts, etc. The first type of direction which could be given to the children is to find "how many piles, with a certain number in each pile, can be made from the objects." The mathematical variability principle should be employed by varying the number in each pile or the total number of objects in each collection. The multiple embodiment principle should be used by changing the type of objects (thus setting a new problem each time) or the type of collection to be formed. The essential aspect is that a collection of objects always can be partitioned into *subcollections* with the *same number* in each and *one other subcollection* with relatively *fewer objects* in it; e.g., a collection of twenty-six objects can be partitioned into four subcollections with six per subcollection *and* two objects in the nonequal set.

One of the first bits of logico-mathematical knowledge the children generally acquire is the sameness of the *number* of objects in the total collection before and after the partitioning process. That is, the child discovers, through his actions, that a pile of objects can always be put back the way it was before the partition (reversibility) and that no objects were added or subtracted; therefore, the number of objects before and after piling is the "same" even though the child *does not* know how many things there are.

Specifically, if a child starts with some objects, makes three piles with six per pile, and one pile of four, the child should know that the total number of objects in the original pile is the same as the number of objects in three piles of six and one pile of four, without knowing there

objects ⟶ partitioning action ⟶
- pile of 6
- pile of 6
- pile of 6
- pile of 4

are 22 objects. The mathematical variability principle (varying the number of objects in each pile) should help the child to the realization that no matter how many are in each pile, the total number in all the piles is equal to the total number of objects. When a child discovers this, he has made a transition from the first learning-instructional phase (exploration) to the second learning-instructional phase (abstraction and representation).

At this point, the operational basis for constructing the concept of a numeration system has been laid. The goal of the second learning-instructional phase is to have the child construct a notational system and construct the place-value concept. Generally, the two-digit numerals are worked on at different age levels than are the three-digit numerals, which in turn are worked on at different age levels than are the four-digit and higher-digit numerals (the generalization of place value).

After the children first enter the second learning-instructional phase they are able to partition a collection into subcollections and know that the number of objects in the original collection is the same as that in all the subcollections. Capitalizing on this knowledge and the ability of the children to engage in rational counting, place-value concepts may be developed. However, until the children can represent any collection of a tens and b ones (where a and b are digits) as "ab", they are not ready to learn the number names for the two-digit numbers and order the numbers from 0 to 100. The number names and the order relation are included in the next learning-instructional phase because the knowledge gained to this point is to be systemized by the children. The basis for the learning which is to take place has been laid in counting out piles of ten. However, the main goal of the formalization-interpretation phase is to, again, systemize the whole numbers from 0 to 100 using the number names. Formalization takes place in the sense that a notational system is developed and organized by the child. The organization of the notational system is based on the abstraction and representation accomplished at the second phase and on the new element of an order relation. The order relation is an essential part of the third learning- instructional phase for the concept of numeration. Without it, the third phase would have little meaning. The order relation, however, is based on one-to-one correspondence, so that preliminary cycles will have to have been completed with regard to one-to-one correspondence and number.

Some Problems

The foregoing model for learning and teaching mathematical concepts needs critical examination in view of other theoretical constructs in an attempt to build the best first approximation of a model possible. That refinements are possible is easily recognized, as a host of theoretical constructs exist which have not yet been integrated into the model. Some of these constructs are figurations and operations, the concept of decentering, and egocentrism of the child. Moreover, a great deal of difference may exist between the model's applicability to mathematical concepts shown to be part of the general development acquisition and those not. Further, no attempt has yet been made to apply the model to geometrical and spatial concepts. It may well be that a model for learning concepts in those areas is greatly different from a model for learning numerical concepts. However, reason does exist that the above model is applicable to geometrical and spatial concepts as Piaget views the grouping structure as a basic structure of mental operations in the infralogical domain. In any case, it is clear that theoretical refinement of the posited model is necessary and that the model needs to be studied across different mathematical concept areas for applicability. Certainly, there are no a priori reasons that a single model is sufficient to account for the learning of disparate mathematical concepts.

The theoretical problems associated with the model should not preclude empirical study of the validity of certain crucial implications of the model. Especially important are such factors as:

1. Are the four factors contributing to the development of certain mathematical concepts critical to the learning of those concepts which have not been shown to be part of the natural course of intellectual development?

2. Can those mathematical concepts not (yet) shown to be developmental phenomena be ordered along the three levels comparable to the concepts known to be part of the developmental process?

3. What is the validity of the learning-instructional strategy involving the mathematical variability and the multiple embodiment principle, in the context of sequencing learning conditions according to the exploration, abstraction-representation, and formalization-interpretation phases.

4. What is validity of learning-instructional phases? Does a structural integration take place only after appropriate overt actions are internalized through reflective abstraction? That is, is the phase of formalization and interpretation identifiable distinct (psychologically) from the phase of abstraction and representation? Reflective abstraction, as a theoretical construct, is appealing, but does it have psychological credibility in mathematics learning?

The experimental studies reported in the following chapters offer a preliminary test of the notion of reflective abstraction in learning of classes and relations by children who had not yet consolidated concrete operational structures. The set of principles inherent in the model described above together with analysis of the relevant genetic structures and mathematical structures were used as guides for the design of the learning materials. No attempt was made to isolate one or more of the underlying factors which may have contributed to resultant learning. The preliminary model was applied in its totality in all of the training studies so that complex interactions of principles in the model were not explored in the learning of selected aspects of classes and relations.

The learning material utilized in the experimental studies did not progress beyond the abstraction and representation phase, except, of course, for cases in which children themselves went into the third phase. The initial learning-instructional tasks were intended to operationally define the concepts for the children. These learning-instructional tasks were written within the constraints of the learning-instructional phase, Exploration. The children were allowed time to engage in free play activities, but direction from the experimenters was included in a highly controlled context to insure that each of the children engaged in overt activities necessary to operationally define the concepts. It was not expected that the children would progress beyond physical knowledge in these initial tasks. The multiple embodiment principle and the mathematical variability principle were employed in such a way that each material set of interest and each concept of interest were included in these initial tasks. A carefully designed schedule for the introduction of terminology was followed throughout the learning materials.

Other learning-instructional tasks were constructed intending to maximize the possibility of children engaging in logical-mathematical experience. These tasks differed from the initial tasks in that they included operations on, or properties of, the concepts involved (e.g., transitivity, asymmetry, class intersection). Hindsight and foresight (anticipation) activities were utilized in the design of these higher-order tasks. No attempts were made to include figural representations or written representations of the higher-order tasks. The tasks always included manipulatable objects. Internal representations of tasks (images or verbal thought) or spoken language were encouraged, but no controls were included to maximize such representation except in the case of terminology developed to communicate the concept elements (e.g., "pair," "partner," "more than"). The time devoted to the two learning-instructional phases was held constant.

In the study by Steffe and Carey (Chapter II) the mathematical structures of equivalence and order relations were used as mathematical models in the construction of the learning materials. The content of the relational structures was length relations defined for open curves of finite length. In the experiment, each child was used as his own control so that information was available prior to the experiment on conservation and transitivity of length relations. The relationship of relational structures to genetic structures has been discussed elsewhere (Steffe, 1973)--similarities and differences pointed out earlier in this chapter were kept in mind, where the similarities were emphasized owing to the status of relations in cognitive development.

The study by Owens (Chapter III) was also concerned with the reflective abstraction and representation of relational structures. Owens, however, employed two relational categories, matching relations and length relations, where each category included equivalence and order relations. Owen's test of whether reflective abstraction took place in the children demanded that the children be able to apply learned properties of matching relations to length relations, where the length relations were only operationally defined. The learning materials, just as in the Steffe and Carey study, employed the foregoing posited model. Owens not only demanded the learned relational properties be transferred to a different relational category in a test for possible reflective abstraction, but he also administered a problem to the children which demanded that transitivity of matching relations be employed in its solution.

In the study conducted by Martin L. Johnson (Chapter IV), rather than attempting to induce properties of equivalence and order relations property by property, children were immersed in total seriation tasks. This decision was predicated on the theory that relational properties emerge as a result of a total scheme of classification or seriation rather than the other way around. It was felt that reflective abstraction would be given the maximal opportunity to operate in a relatively short period of time (16 instructional days out of 22 consecutive instructional days) when content shown to be developmental phenomena.

Lesh, in the study reported in Chapter V, did not begin with mathematical structures in his learning program but, rather, conducted a pilot investigation using the genetic structures as his starting point. On the basis of the genetic structures, Lesh generated three sequences of tasks, one dealing with seriation, one with number, and one with classification. The tasks were subjected to empirical validation as to their sequential nature and difficulty. A transfer of learning experiment was then carried out where training was given on seriation and classification and transfer to the number tasks tested. The preliminary model presented above was utilized in the construction of the learning materials for the transfer of learning experiment.

A fundamental issue raised by Lesh's experiment was mentioned in discussion of the El'konin-Davydov experiment. The mental operations Lesh worked with were taken from the genetic structures. Because Lesh was dealing with such fundamental concepts, it was possible for him to proceed as he did. However, Piagetian theory has less to offer concerning possible genetic structures underlying more advanced concepts, and even if mathematical structures are considered, mental operations underpinning these structures may be different from those underpinning genetic structures. In that the results of Lesh's training study are positive, more experiments need to be conducted designed to shed light on mental operations underlying the child's concept of number.

The study by David C. Johnson (Chapter VI) was concerned basically with educational technology. No attempt was made to experimentally determine basic mental operations underlying the child's concept of number. Rather, it was assumed that Piagetian theory was essentially correct as it is concerned with relations and classification. The experimenter was explicitly aware of structural differences in mathematical structures and genetic structures dealing with classes and relations, but emphasized the similarities rather than the differences.

Of the experiments reported in this monograph, David C. Johnson's experiment provides the best test of the construct "reflective abstraction" in the case of learning concepts shown to be developmental phenomena. In the learning material, children were given definite opportunities to engage in mathematical as well as physical experiences. The transfer tests all demanded mathematical knowledge for successful completion, whereas the achievement measures demanded only physical knowledge. So, a test was possible of the amount of mathematical knowledge the learning material produced in the children

The study conducted by Carpenter (Chapter VII) was concerned with development of mental operations regarding measurement. The study was not experimental in nature but was related to the previous studies in the monograph by virtue of the content of the tasks, the variables controlled, and its developmental nature. In particular, Carpenter tested the degree to which young children possess the logical structures necessary to identify some factors involved in the development of measurement concepts. The study provides, within a scope limited by the tasks and factors studied, baseline data for designing experiments relevant to specific questions implied by the model.

MAP Publications--September 1975 complete Update

Research Reports

1) Discovering Psychological Principles for Mathematics Instruction
Charles D. Smock

2) The Use of Mathematical Models for Describing Sequential Search Strategies in Elementary School Children
Robert Lissitz, Philip Lewis, Patrick Brady

3) Measures of Language Comprehension: Relations to Performance
Rodney Cocking

4) Models for Performance in Syllogistic Reasoning
Paul Silverman

5) Elementary and Middle School Children's Comprehension of Euclidean Transformations
Richard Kidder

6) Ethology's Modest Contribution to the Assessment of Education
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7) When Do We Know What a Child Knows?
Irving Sigel

8) What Should We Measure When We Measure Linguistic Competence?
Joan Baratz

9) New Perspectives in Compensatory Education
Glendon Nimnicht

10) An Analysis of Organizational Structure by Application of Individual Differences Multidimensional Scaling
Catherine L. Jones, Robert Lissitz, Philip Lewis, Charles Smock

11) A Survey of Cognitive Abilities of Children in Different School Programs
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12) Lifeways and Cognitive Performance in Australia and the United States
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13) Development of Concrete Operations in Urban and Rural Communities: A Study of the Effect of the Mathemagenic Activities Program
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14) Epistemology and Education (A Collection of Essays)
Charles D. Smock, Ernst von Glasersfeld

15) Development of Social Relations: Role-Taking Skills and Classification Abilities
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Research Reports

16) Radical Constructivism & Piaget's Concept of Knowledge
E. von Glasersfeld, June 1975

17) The Concepts of Appearance, Disappearance, and Reappearance: A Developmental Semantic Analysis
Michael Tomasello and Ernst von Glasersfeld, August 1975

18) On a Model for Learning and Teaching Mathematics
Leslie P. Steffe and Charles D. Smock, June 1976

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Piaget's Thinking
about the
Development of Space Concepts and Geometry[1]

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Preface

The Mathemagenic Activities Program is a model for early childhood education, sponsored by the National Follow Through Program, based primarily on Piaget's theory of cognitive development. The emphasis on structure of cognitive operational systems (logico-mathematical experiences) as the basis for understanding intellectual development provides mathematics educators with a unique opportunity for potential contributions to child development. This paper explores some of the implications of Piaget's theory for mathematics learning and teaching.

Charles D. Smock, Director

Piaget's Thinking
about the
Development of Space Concepts and Geometry

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> For the genetic epistemologist, knowledge results from continuous construction, since in each act of understanding, some degree of invention is involved . . .
>
> Jean Piaget

The conceptual revolution for psychology and education required by Piaget's epistemology is, I think, more appreciated than understood. That is, Piagetian ideas pass around as common currency, among researchers and practitioners alike, but all too often turn out to be only vague facsimiles or counterfeit copies. If we are to benefit fully from Piagetian ideas of psychological development and learning of mathematical concepts, we must translate those ideas with as little transformation as possible. In this paper, I hope to do just that in the context of the purpose of this workshop, i.e., review selected theoretical and methodological issues relevant to research into the development and learning of space and geometry concepts. Specifically, I will (a) review certain critical features of the epistemological and theoretical aspects of Piaget's position vis-á-vis the development of space concepts, (b) review the available evidence concerning the construction of the "permanent object" which is the fundamental invariant of our spatial world, (c) summarize Piaget's early work on space and geometry, and (d) offer some methodological suggestions and guidelines for inquiring into cognitive development in general and space and geometry in particular.

First, as I argued in an earlier paper (Smock, 1973) a coherent, accumulative body of "facts" based on psychological research on the development of space is necessary but not sufficient for building a theory of instruction. Our observation and interpretation of facts are heavily prestructured by our epistemological belief and knowledge base or cognitive structures. Understanding of the relevance and implications of Piagetian concepts and data for education depends on each of us starting at the beginning--i.e., with his theory of the nature of knowledge and knowledge acquisition and then building up our personal knowledge base of empirical findings and demonstration of ideas generated from appropriate experimental settings. Until recently, American researchers' contribution to an accumulative body of knowledge relevant to Piaget's theory was limited to replication of Genevan studies. Current research indicates that the methodological implications (of which I will speak later) of the theory are more clearly recognized (e.g., Forman, 1973; Overton & Jackson, 1973; Toussaint, 1974). But, in any case, the mathematics education researcher needs to build his understanding of the implications of psychological research findings on a firm understanding of the epistemological consideration that is the foundation of Piaget's theory of cognitive development and learning.

Piaget is characterized as a "natural genetic epistemologist." The adjectives may be interpreted with little ambiguity, i.e., "observations of the origins" (of knowledge). However, the specific epistemological position of Piaget is not so readily located, even from his own writings, but there are strong indications that he represents a "radical constructivist" view (Smock, 1973; Smock and von Glasersfeld, 1974; von Glasersfeld, 1975). That is, the environment is, and must remain, a "black box." All we can ever "know" is our own cognitive structures. Knowledge is no more, and no less, than constructed invariants of organism-environment relations, but this construction

> involves the adjustment of, for instance, percepts to conceptual structures which the perceiver has already assembled; and this adjustment of the new to the old is called 'assimilation.' But cognitive equilibration also involves the adjustment of concepts to percepts, and this second type of adjustment, which can take the form of creating a novel structure or of combining several already assembled structures to form a larger conceptual unit, is called 'accommodation.' (von Glasersfeld, 1974, p.4).

The process of development and learning (cognitive reorganization) is, then, a response to conflict among internal functional structures (Smock, 1969, 1974).

The genetic epistemology of Piaget assumes then, that spatial concepts are constructed through commerce with the perceived environment and is only one aspect of the development of general cognitive structures. Expectedly, then, three themes characterize the Genevan research on space. First, the primary concern is on conceptual and representational space not perceptual space. Thus, acquisition of concepts of the spatial world(s) is a product of general intellectual development. The genesis of space perception is used to highlight the contrasts of perception with that of space representations.

A second theme emphasizes that spatial representations are built up through the process of organization of actions and/or logico-mathematical experience. Initially these are sensory-motor actions (resulting in "practical" space) which later are internalized actions and culminate in operational (intra-logical) systems. The active manipulation of the spatial-temporal environment (objects, empty space, intervals, duration, movements), rather than a passive copying by the perceptual system, generates representations of space.

Finally, a third theme is the characterization of spatial concept acquisitions according to the type of geometric concepts involved--topological, projective, or Euclidean. According to Piaget the historical order (Euclidean, projective, topological) is reversed logically and ontogenetically.

Object Permanence

The construction of *space representation* emerges at the beginning of the concrete operational period (ages 5-7) from preceding preoperational cognitive acquisitions of the "practical group of displacement" and the even earlier construction of the sensori-motor period. In the beginning, there is an "object"-- the origin of, and basic unit for, the development of spatial relations as well as the starting point of Piaget's analyses of the nature of knowledge.[2] Thus, an understanding of the development of object permanence is essential for "knowing" Piaget's epistemology as well as his psychological theory of the concepts of space.[3]

The infant is at first not aware of any permanent objects, but merely "perceptual pictures", which appear, disappear, and perhaps reappear. In its simplest form, attainment of object permanence means that the infant *knows* the object continues to exist when it is outside of the perceptual field. The indicator of this knowledge (object permanence) consists in "true search" on the subject's part. A *true search* is a search for the vanished object independent of subject's on-going actions and perceptions.

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[2] Citations of Piaget's work will be used only as necessary to identify specific sources of ideas (e.g., quotations). A selective set of primary sources is listed in the references.

[3] See E. von Glasersfeld's (1974) discussion for more details on the epistemological aspects of the notion of the constructed permanent "object."

The simplest procedure Piaget used to test the "object permanence" is as follows:

1. E shows the child the object (e.g., a doll).
2. E grasps the object in his hand so that the child no longer sees the object.
3. E puts his hand under a coverlet.
4. E withdraws the hand closed.
5. E reaches the closed hand to the child.

If the child opens the hand, finds it empty, goes on to search for the object under the coverlet and gets it, then he is considered to have attained the concept of object (i.e., object permanence).

The "logic" supposedly necessary for the attainment of object permanence by the child is as follows:

1. There is an object.
2. A and B are the only possible places the object could be.
3. The object is not at A.
4. Therefore, the object must be at B.

According to Piaget, the infant's conception of external objects as permanent, independent entities is acquired in six distinct stages during the sensory motor period (0-2 years). Evidence for the stages was based primarily on Piaget's observation of his three children's reactions to objects which disappear from view. The behavior pattern characteristics of each stage, along with the estimated age ranges reported by Piaget, are summarized below.

During Stage I and II the infant has the ability for recognition, has intercoordination of schemata, and shows simple expectation. However, the ability to recognize the mother's face, or to look at the object from which the sound comes from, or to continue to look at the place where the object has just vanished and to continue with the sucking response after removal of the nipple, is not a manifestation of the attainment of object permanence.

Recognition

Conceptually, recognition of an object means that the S knows that object he sees now is the same object that he saw previously. Recognition is operationally identified by the fact the subject can respond in the same way when the same object is presented at two time points. For Piaget, recognition is at first only a particular instance of assimilation: The thing recognized stimulates and feeds the sensorimotor schema which was previously constructed for its use. In order that the recognized representation or "picture" becomes an externalized "object," it must be dissociated from the action itself and the causal relations dependent on the immediate activity.

Intercoordination of Schemata

From the second month of life and the beginning of the third, the child tries to look at the objects he hears, thus revealing the relationships being established between sounds and visual pictures. Does this mean that by presenting certain sounds, the anticipation of a certain image of an object is elicited and thus that the child has already an object concept? Piaget argues that simple intercoordination of schemata between sight and hearing, at the outset, does not generate an objective identity of the visual image and auditory image, but simply a subjective identity, i.e., the child tries to see what he hears because each schema of assimilation seeks to encompass the whole universe. Discovery of the visual picture announced by the sound is only the extension of

101

the act of trying to see. It is not the same case as that of an adult when his act of searching with the glance is accompanied by a belief in the firm existence of the object looked at.

Simple Expectation (Anticipation)

True search is an indication of the beginning of the object concept but, again, simple expectation is not. Simple expectation refers to those behavior patterns in which the search for vanished objects is only a continuation of the earlier act of accommodation. The child only preserves the orientation generated by the earlier perception, e.g., in the case of the disappearing visual image, the child limits himself to looking at the place where the object vanished. If nothing reappears, he quickly shifts attention.

In simple expectation, the vanished object is not yet for the child a permanent object which has been moved, it is a mere image which re-enters the void as soon as it vanishes, and emerges from it for no objective reason. True search is an active search and includes intervention of movements which do not solely extend the interrupted action. In this case, the child will find out where the "thing" could have been put, he will remove obstacles, change the position of the presenting objects at hand, and so on.

One could, perhaps, argue that the child's failure to engage in active search (i.e., simple expectation) is due to lack of motor skill at this early age. However, if the child, while not knowing how to search (motorically) for the absent object, nevertheless believes in its permanence, then "true search" should begin as soon as prehension skills have been acquired. Such is not the case as the child's behavior in Stage III indicates.

Stage III is a transition period from prehension of an object at hand to true search for a missing object. In this intermediate stage, five types of behavior are distinguishable: (1) visual accommodation to rapid movements; (2) interrupted prehension, (3) deferred circular reaction, (4) the reconstruction of an invisible whole from a visible fraction, and (5) the removal of the obstacle preventing perception. The first of these behavior patterns merely extends those of the second stage, and the fifth fulfills those of the fourth stage.

All the behavior patterns enumerated hitherto merely extend the action in progress. Clearly, visual accommodations to rapid movements, interrupted prehension, and deferred circular reactions, all consist merely in returning to the momentarily suspended act--not in complicating that action by removing the obstacles which arise. The reconstruction of invisible totalities and the removal of obstacles preventing perception both seem to involve such differentiation, but this only appears to be true. That is, when the child tries to get to a half-hidden object and, to do this, removes the obstacles which cover the hidden portion, the action is by no means one as complicated as that of removing a screen masking the entire object. In the latter case the child must momentarily give up the attempt at direct prehension of the subject in order to raise a screen which is recognized as such. On the contrary, in the former case, the child sees part of the object and tries to grasp it; he only reconstructs the totality as a function of this on-going action and not because of new action pattern, consisting of removing the screen, has been differentiated. Removal of obstacles preventing perception requires knowledge of an object in relation to the subject and not in relation to the object: i.e., the obstacle-screen and the object as such are not yet related. From this point of view, the object is still only the extension of the action in progress.

True Search

Stage IV marks an important transition. Prior to Stage IV, the infant lacks object permanence and knows a thing and its location only in the context of his on-going actions. He either cannot find hidden things or can only find them when he has begun to reach for them before they disappear. In Stage IV, the infant is aware of the object's permanence. If he observes an object disappear, he searches for it even when he has not begun to reach for the object before it disappeared. However, his objectivity has an important limitation which indicates that his ideas of things continue to be bound up with how he acts upon them. Objects are not localized in terms of where it has moved, but in terms of where it was found in the past.

Piaget employed the following procedure to test children for Stage IV behavior. First an object was hidden in place A, and then in place B. Piaget noted that after the child had found the object at A, they attentively watched the object being hidden at B. However, when the object disappeared in view, the children searched at A. This pattern of success at A and failure at B was referred to as $A\bar{B}$.

Three possible explanations are offered for the $A\bar{B}$ behavior: difficulties in memory, in spatial localization, and in object conceptualization. Piaget (1954) makes it clear however, that the first two are considered but different aspects of the difficulty in object conceptualization: "Faced with the disappearance of the object, the child immediately ceases to reflect: in other words, he does not try to remember the sequence of positions, and thus merely returns to the place where he was successful the first time. p. 61". The memory explanation is simply a way of saying the child has trouble keeping track of places, and the second, or spatial localization explanation, provides reason for this difficulty. Further, Piaget argues that the infant localizes objects in terms of a scheme based upon the infant's prior actions. Accordingly, the A$\bar{B}$ error occurs because there has been no previous action at B, and therefore the infant has no "memory" of that place. On the other hand, he is able to localize A because it was at that locale that his practical action brought him a toy. When he sees a toy being hidden at B, he registers only that a toy is being hidden at a "place" and so searches at the only place he is able to localize.

When talking about the third explanation, Piaget points out that adults are able to think of particular objects only because of the assumption that objects are independent of the many places they may occupy. However, if we did not distinguish thing from place, the adult would be aware of "ball-under-the-arm-chair," "ball-under-the-cushion,", etc., which is what the infant does in Stage IV. The child endows objects with only a few special positions without being able, consequently, to consider it as entirely independent of them. "In a general way," Piaget (1954) concludes, "in all the observations in which the child searches for A for what he has seen disappear in B, the explanation should be sought in the fact that the object is not yet sufficiently individualized to be dissociated from the global behavior related to position A (p. 63)."

The first acquisition of Stage V is signified by success on a task where an object is hidden under a (first) screen which the child finds then under a second screen, but only under the second one. As mentioned above, the child succeeds in keeping track of only the visible displacements of the object and locates it only when he has actually seen it. From repeated experiments, Piaget found that when an invisible displacement of object intervenes, the child relapses into the same difficulties which he has already overcome when visible displacements were involved. These findings furnish us with a good example of the law of "temporal displacement," i.e., when an operation passes over from one plane of action to another, it has to be relearned on this new plane. In particular, the groups of displacements of object which, at the beginning of the fifth stage, had been constituted on the plane of direct perception of relationships of position, must be formed anew as soon as it has been transferred to the plane of representation of these relationships.

In the final (VI) stage, the object is not as it was during the first four stages, i.e., merely an extension of various accommodations. Nor is it, as in Stage V, merely a permanent body in motion whose movements have become independent of self but only to the extent and scope of perception. Instead, in Stage VI the object is definitely now freed from perception and practical action.

It should be emphasized here, that searching for an object under a screen when the subject has seen it disappear there (Stage IV & V) does not necessarily presuppose that the subject "imagines" the object under the screen but simply that he had understood the relation of the two objects at the moment he perceived it (at the moment when the object was covered) and, therefore, interprets the screen as a sign of the actual presence of the object. It is one thing to assume the permanence of an object when one has just seen it or when some other object in sight recalls its presence. It is quite another thing to imagine the first object when there are no perceptual signs to confirm its hidden existence. True representation begins only when no perceived sign commands belief in permanency; that is, from the moment when the vanished object is displaced according to an itinerary which the subject may deduce but not perceive. With regard to the infant at the fifth stage the objects are not "permanent" to the extent that he does not know how to imagine or to deduce the invisible displacements of bodies as objects truly independent of the self. On the contrary, the representation and deduction characteristics of the sixth stage result in disassociating the object from action and perception and objects in motion become real objects independent of the self.

Review of Recent Studies

A review of studies of object permanence revealed that most are concentrated on Stage IV, because Stage IV is considered by Piaget to be the critical period where the "true search" behavior begins. The procedures used in these studies

107

generally are the same as those Piaget used except for additional control of critical variables such as the length of interval between hiding the object and permitting the child to search (usually a 3-second delay was used), and the number of times the object is hidden at A before it is hidden at B (usually the object was hidden twice at A before it is hidden at B).

Gratch and Landers (1971) observed infants biweekly until they were between 6 and 12 months of age in order to replicate and elaborate on the observations Piaget employed to define Stage IV of object concept. Their observations suggest that the Stage IV phenomenon is part of an age-related sequence of responses to hidden objects. Before the infants found an object hidden out of their reach, they were able to find it when partially hidden (P) and were able to find objects that were grasped before they were covered (G). The Stage IV error ($A\bar{B}$) phenomenon occurred repeatedly over a period of 1-3 months. During that period, however, the character of $A\bar{B}$ changed. Initially, Ss seemed to ignore the displacement of the object to a second position. Later, they appeared to be in conflict over whether to use the cues provided by their prior successful searches or their awareness of the displacement of the object. Finally, they came to rely on the cue of object displacement.

Landers (1971) studied the effects of "active experience" (i.e., searching for the hidden toy) versus "passive experience" (i.e., observing but not searching) on the $A\bar{B}$ behavior.[4] Using infants between the ages of 7 1/2 to 10 1/2 months, he found that infants who had more experience finding an object at the A position tended to have more difficulty finding the object when it was hidden at the B position. This suggests that simply watching the E hide and uncover a toy at A does not establish the A side as a "special" place, which active search does. The results seem to clarify and establish empirically Piaget's arguments that the Stage IV infant's behavior and object concepts are dependent upon "context" and "action." The previously reinforced motor response seems to be a more potent aid to memory than the most recent visual input when representational processes are just beginning.

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[4] Three groups were used: (GI)-"low active A-experience group"-two active experiences at A; (GII)-"high-active A-experience group"-10 active searchings at A; (GIII)-"high-passive A-experience group"-two searching trials at A plus 8 observing trials.

Evans and Gratch (1972) studied the $A\bar{B}$ error to evaluate the relative merits of place (spatial localization) and thing-of-place (object conceptualization) arguments Piaget offered for explaining the $A\bar{B}$ error. Infants between the age of 8 1/2 and 10 1/2 months served as subjects. Half of the subjects were assigned to "same toy" condition and the other half to "toy change" condition in which different toys were used on B-place test trials than those used on the A-place practice trials. The result showed no difference between Ss' performance on the two conditions. This result, Evans and Gratch concluded, supported the notion that the $A\bar{B}$ error occurs because A has somehow become "a place where hidden toys are found", rather than because, as Piaget has argued, the particular object belongs at A. If Piaget's thing-of-place argument were correct, then the fact that the child sees a new object hidden at B should increase the probability that he will search correctly since no previous action by the child has endowed the new object with a place.

Lecompte and Gratch (1972) investigated directly the development of object identity in infants by varying the object instead of its spatial position. This involves tricking the child by hiding one thing and having him find another. The assumption is that if a child is aware of the permanent nature of the object being hidden, in the sense in which adults are and which Piaget attributes to infants at Stage IV, he will be surprised when he is tricked because his belief in permanence will have been violated. Ss consisted of infants at three different age levels: 9, 12 and 18 months. Two rating scales, both with seven categories were devised to evaluate the child's immediate reaction to trick (levels of puzzlement) and his subsequent behavior (what he was surprised or puzzled about). The result showed that infants at all age levels reacted differently on the trick trials than they did either before or after the trick (two trick trials were inserted among a total of nine trials). Older infants reacted with high puzzlement and searched for the missing toy and for the causes of the disappearance. Younger infants were mildly puzzled and only focused on the new toy. The authors concluded that these results conformed to Piaget's account of the development of the object concept.

As has been mentioned previously, Piaget has not controlled the time interval between object being hidden at B and the time S is allowed to search. Harris (1973) studied the role of delay and compared the Ss' performances on a 0-sec and 5-sec delay task. Infants 10 months of age served as Ss. The results indicated that infants of this age search correctly, but errors are more likely with delay and if cues previously associated with finding the object distract the infant after its disappearance. Harris draws attention to the similarity between this behavior of the human infant and the frontally lesioned primate. He proposes that maturation of frontal cortex may be important for the development of search behavior. The argument is based on the fact that Harlow, Harlow, Rueping & Mason (1960) found that prior to 5 months, training on delayed-response tasks led to little or no improvement. After 5 months, learning was increasingly more rapid with increasing age. The learning curves for 0 and 5-sec tasks were similar, but the ability to solve the delay task developed more slowly. Harris (1973) concludes that these data demonstrate that the infant's immaturity makes him susceptible to proactive interference in short-term memory.

Webb, Massar, and Nadolny (1972) observed the behavior of 14 and 16 month-old children in searching for hidden objects. A three-choice, instead of two-choice, task was used and the subjects were allowed to search for the object on each trial until it was found. The major finding was that the second-choice behavior of the 14-month-old Ss was essentially random but that of the 16-month-old children was predominately correct after an initial error. The interpretation of these authors was that the child knows the location of the object in two distinct and relatively independent ways. That is, he remembers both past actions in locating the object, and the location of where he saw the object hidden. In the search, first credence is given to the past location of the object whereas the immediately prior perception becomes functional only after the other cues prove unreliable. Thus, the child knows the correct location of the hidden object in some sense even while making the error on the initial choice. The interpretation applied to these data has an interesting analogy to several recent attempts to reformulate Piaget's conservation problems; i.e., that perceptual strategies interfere with the child's use of other solution processes. In the present report Webb, et al.

suggest that an overdependence on action-marked cues and past success is seen as overriding a functioning memory for visually presented events.

Gruber, Girgus and Bannasisi (1971) modified Piaget's methods of studying object permanence in children in order to study development of object concepts in cats. Eight behavioral tasks were constructed: 1. auditory stimulus (click) off to one side; 2. an object swung in circle around the kitten; 3. object placed in front of kitten and then moved slightly; 4. object and kitten placed on stool; 5. kitten playing with object and auditory distraction introduced; 6. kitten playing with object and visual distraction introduced; 7. kitten is playing with object and object is covered while kitten is distracted; 8. object is covered while kitten is playing with it. The data indicated that cats reach only an early "developmental" limit. Unlike children, the kittens were unable to follow an object through a series of invisible displacements. Interestingly, house-reared cats showed similar limitations but advanced more rapidly than cage-reared animals. Finally, the longitudinal data suggested that cats go through four, rather than the six, stages found in children. Kittens develop in 24 weeks as far as children do in the first year, but the child's behavior eventually becomes more complex and more general.

Uzgiris and Hunt (1966) constructed a series of scales for assessing infant psychological development. For each of the series of "eliciting situations," certain infant actions were selected as indicative of significant steps in that branch of psychological development.[5] To give a concrete idea of the scales, the "noticing the disappearance of a slowly moving object" item is a good example:

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[5] Escalona and Corman (1967) also constructed a scale for assessing the degree of object permanence. The administration condition and responses are described which enabled the examiner to score the child as belonging to a particular level of a particular stage.

Noticing the disappearance of a slowly moving object

Location: The infant may be supine on a flat surface, in an infant seat, or sitting up by himself.

Object: Any bright object that attracts the infant's attention:

Directions: Once the infant has focused on the object, move it slowly to one side and away from the infant, making it disappear below the edge of the infant's seat or the surface on which he is placed. After a few moments bring the object back in front and slightly above the infant's eyes from the opposite site. Always move the object in the same direction and have it disappear at the same point.

Repeat: 3-4 times.

Infant Actions:

a. Does not follow object to point of disappearance.

b. Looses interest as soon as object disappears (eyes begin to wander and then focus on any interesting object within view).

*c. Lingers with glance at the point where the object has disappeared.

*d. After several presentations, returns glance to the starting point or the point of reappearance before the object has reappeared.

e. Searches with eyes around the point where the object has disappeared.

*Actions marked with an asterisk (*) are considered critical for this particular step of development.

Miller, Cohen and Hill (1970) replicated the ordinal scales of object permanence constructed by Uzgiris and Hunt with 84 infants (ages 6, 8, 10, 12, 14, 16, and 18 months). Consistent age changes were found that suggested two overlapping developmental dimensions: 1) the ability to deal with visible versus invisible displacements; and, 2) ability to handle nonsequential versus sequential displacements. However, the trend of data suggested, contrary to previous evidence, that the infant can cope with single invisible displacements not involving movement before he can handle complex visible displacements that do involve movement.

Bell (1970) studied the development of the concept of object as related to infant-mother attachment with infants between the ages of 8½ and 11 months. Two scales, closely comparable to those constructed by Escalona and Corman (1967) and Uzgiris and Hunt (1966) were devised. Seventy percent of the infants showed "positive decalage"; i.e., tended to be more advanced in the development of "person" permanence than in the development of the concept of inanimate objects as permanent. Another important finding was that the development of the object concept was intimately associated with the attachment of a baby to his mother. The babies who manifest strong evidence of "decalage" displayed more active efforts to establish and maintain interaction with the mother through approaching, reaching and/or initiating interaction.

Bower (1971) has raised the question as to whether object permanence could be a built-in property of the nervous system (as is the case with so many other kinds of perceptual knowledge). During the experiment with 20, 40 and 80 day old infants a screen moved in from one side and covered the object. After various intervals (1.5, 3, 7.5 or 15 sec) the screen was moved away. In half of the trials the object remained as the screen moved away and in the other trials the object was removed prior to removing the screen. The results showed that when the object had been occluded for 1.5 sec, all the infants manifested greater surprise at its non-reappearance than at its reappearance. The index of surprise here was determined by a change in heart rate. Thus, it would seem that the infants expected the object to be present. However, while the oldest infants "expected" the object to reappear (i.e., showed the greatest heart rate decelleration) after the longest occlusion period, the youngest infants exhibited a reverse effect after the longest occlusion period; i.e., showed more surprise at the object reappearance than at its non-reappearance. It seems that even very young infants know that an object is still there (built-in structure) after it has been hidden, but if the time of occlusion is prolonged, they forget the object altogether.

Bower (1971) suggested that while the older infants identify an object by its features, the younger infants (less than 16 weeks) identify objects by place or movement. To test this idea, two groups of infants - the younger infants were between 6-16 weeks and the older ones were between 16-22 weeks - were examined in four situations: (1) An object moved along a track, went behind a screen, emerged on the other side, moved on for a short distance, stopped and then returned to its original position. (2) The object moved along the track, went behind the screen and at the moment when the object should have emerged on the other side of the screen a totally different object emerged, moved on for a short distance before reversing and repeating the entire cycle in the opposite direction. (3) The object moved along a track as before, except at the time when according to its speed before occlusion it should still have been behind the screen, an identical object moved out. (4) The object moved along the track as before, and at the time when it should have been behind the screen a totally different object moved out. Table 1 summarizes the results:

Table 1

Infant's Reactions to Bower's Object-Movement Situations

	Conditions	Younger Ss (6-16 Weeks)	Older Ss (16-22 Weeks)
(1)	One object, one movement	Continued to follow the path of movement when the object stopped	Stopped tracking the object when it stopped
(2)	Two objects, one movement	(Same response as above)	25% of the times Ss would look to the other side of the screen when the object stopped
(3)	One object, two movements	Upset and refused to look any more; Ss did not continue to follow the object's path when it stopped	On every trial, Ss looked to the other side of the track when the object stopped
(4)	Two objects, two movements	(Same response as above)	(Same response as above)

From these results, Bower argues that the younger Ss responded not to moving objects but to movement per se. Similarly, their responses were not to stationery objects, but to places. On the contrary, older Ss identify an object by its features rather than by its place or movements; for them different features imply different objects that can move independently.

Conclusion

Object permanence means that an infant knows the object continues to exist when it is outside of the perceptual and action field. "True search" is the indication of object permanence. By true search Piaget means that the subject engages in an active search for the vanished object independent of his action and perception.

Six stages in the development of the concept of object permanence has been identified with Stage IV marking an important transition. Prior to Stage IV, the infant knows a thing only in the context of his ongoing actions; he is concerned with his objective as one and the same thing as his desire. In short, before Stage IV the action is the source of external world. After Stage IV the object becomes detached from the *S*'s activity and gradually acquires an independent status.

Recent studies of object permanence have concentrated on the infant's Stage IV concept of object and, in particular, on the A$\bar{B}$ phenomenon. A$\bar{B}$ means that after the *S* has experience of successful searches at place A, though he has watched the object being displaced to place B, his search for the object is at A. The implication is, that the infant conceives the object as *being* the product of his action: the toy does not exist by itself, but is the end result of the reaching of his hand at A. At the end of Stage IV, the infant's search for object begins to oscillate between place A and B, and this is the mark of the beginning of the object being detached from the action. Studies of infant object concept at Stage IV generally have confirmed the developmental sequence observed and theoretical interpretations reported by Piaget.

Landers' (1971) study showed the influence of prior motor experience on the subsequent search behavior of the infant. Evan and Gratch (1972) argue against Piaget's thing-of-place interpretation of A$\bar{B}$ behavior but do not take the role of action into consideration. The "thing-of-place-A" concept of object that Piaget offered is simply the result of the "action-at-place-A". If the subject has only had the experience of "action-at-A" there is no reason to expect that the infant would search at B even when the infant has noticed the object being hidden at B is different from what was hidden at A in the former trials.

The critical point in Lacompte and Gratch's (1972) study is that while the older Ss were engaging in a search for the missing toy, the younger Ss were only focusing on the new toy. The younger Ss' behavior only indicated that the "toy change" had been noticed, or, the ability to identify the object by its features had already been demonstrated. Bower's (1971) study showed that the ability to identify the object by its features was present at the age of 16-22 weeks. Since this behavior does not fulfill the criterion of "true search" there is no reason to believe the object still exists for the infant once it vanishes from the S's perceptual field.

Harris (1973) suggests a physiological basis for lack of object permanence. For him the inability to solve delayed reaction tasks is the basic source of difficulty for infants. It might be possible, as he indicates, that the ability to solve such delay tasks is related to maturation of frontal cortex.

Webb, et al.(1972) used a new procedure, in which the infants were allowed to correct errors and make a second choice on a three-choice task. The results of the study showed predominant correct second choice of the 16-month-old Ss. It is clear that 16-month-old infants do not forget prior relevant perceptual input; the error on the first attempt only shows that, at this age, "action" overrides prior perceptions at this cognitive developmental stage. It seems, therefore, that the development of any conservation (object, area, weight, etc.) goes through the same process with action being predominant initially, then perceptual factors and finally the representational (logico-mathematical) system.

Gruber et al. (1971) demonstrated that cats never go beyond the Stage IV concept of object. This result seems to suggest again that there is a corresponding change in nervous system (e.g., in frontal cortex) along with the development of object concepts.

The scales that Uzgiris and Hunt (1966) and Escalona and Corman (1967) developed to standardize tests of infant's object concept should prove valuable. Piaget always allows great latitude of procedures in experiments and in many cases it is the best way to find out about what a child knows. However, standardization of procedures are, at least, necessary for comparative analyses.

Bell (1970) found that the development of person permanence was more advanced than that of inanimate object concepts. Though Bell did not emphasize the role of experience in the development of object permanence, infants who had more interaction with the mother were significantly more advanced in the development of person permanence than those with less interaction. Piaget (1959), in interpreting the transition from Stage IV to Stage V, emphasizes the role of such experience.

Construction of Space

The analysis of the nature of space has pre-occupied philosophers and scientists for centuries. Piaget (1948) was intrigued by the fact that the historical and logical sequences of geometry (measurement of space) were in conflict. Geometry primers are almost unanimous in presenting the fundamental ideas of space as resting upon Euclidean concepts such as straight lines, angles, squares, circles, measurements, and the like. On the other hand, abstract geometrical analysis tends to show that the fundamental spatial concepts are not Euclidean at all, but 'topological'; i.e., based entirely on qualitative correspondence involving concepts like proximity and separation, order and enclosure. With his studies of the child's conception of space, Piaget has successfully demonstrated that the child's space invariably begins with simple (basic) topological types of relationships long before it becomes projective or Euclidean in nature.

The misconception that space concepts begin with simple Euclidean characteristics arises primarily from ignoring the fact that the evolution of spatial relations proceeds at two different levels. There is an interval of several years between perceptual and conceptual construction of space (despite their pursuing a similar path of development). The children's perceptual space has reached projective and quasi-metric levels during the first year of life, when their conceptual space has barely begun.

The present review concerns only conceptual space. But since the perceptual and sensori-motor structures constitute both the point of departure and the foundation of the entire conceptual construction of space, we will start by going over briefly the development of perceptual space as seen in the child's perception of shape.

In the experiments of the child's perception of shapes, Piaget (1948) identified three stages. During Stage I the only shapes recognized, and drawn, are closed, rounded shapes and those based on simple topological relations such as openness or closure, proximity and separation, surrounding, etc. These relations express the simplest possible coordinations of actions; e.g., following a contour step by step, surrounding, separating, and so on. With Stage II the recognition of Euclidean shapes begins based on the distinction between straight and curved lines, angles of different sizes, paralleles, and relations between equal or unequal sides of figures. At this level, the coordinations of actions is of a more complex type in that the child now recognizes a straight line by the action of following - with hand or eye - without changing direction, and recognizes an angle by two such intersecting movements. Lastly, at Stage III the child is able to return systematically to a fixed point of reference while exploring an object. That is, he now can coordinate all his movements into a single whole according to a system of reference. For example, if the object is a figure of a starfish engraved on a surface, and the child is asked to identify the figure through haptic exploration, a child at Stage III is able to touch each arm in turn, exploring the angles between the arms, and returning systematically to the center where the arms meet.

Perceptual organization of space proceeds through a developmental stage sequence. At first it is based on topological relationships, and later on projective and Euclidean relations. Does conceptual space, after some year's interval, pass through the same phases?

The most important difference between topological relations and the projective and Euclidean relations is in the way in which different figures or objects are related to one another. Psychologically, topological relations are the most primitive ones; the relations of proximity, separation, order, enclosure and continuity are built up empirically between the various parts of figures or patterns which they organize. These relations are independent of any contraction or expansion of the figures and, therefore, do not conserve features such as distances, straight lines, angles, etc., during changes of shape. Hence we may say that topological space is purely internal to a particular figure whose intrinsic properties it expresses, and it is impossible for relationships of this type to lead to comprehensive systems linking different figures together. It is in this sense that topological relationships are considered primitive.

Projective space begins psychologically at the point when the object or pattern is no longer viewed in isolation, but begins to be considered in relation to a 'point of view'; the viewpoint of the subject (in which case a perspective relationship is involved) or that of other objects on which it is projected. Because of this property concerning viewpoints, the study of projective space can be called 'geometry of viewpoints'.

Euclidean space is different from projective space in that the concepts of distance and measurement are introduced. It deals with the orientation of objects relative to each other and to a system of reference points arranged along different dimensions. Because of its concern with the objects as such, the study of Euclidean space may be called 'geometry of objects'.

In short, topological space deals with the internal relations of the isolated object, projective space deals with relations of objects to the subject and Euclidean space deals with relations of objects to objects.

The evolution of the conceptual spaces can be clearly demonstrated in children's drawings of geometrical figures. Three distinct stages can be identified from the drawings of children when they are asked to copy figures. During Stage I (0-4 years), the circle is drawn as an irregular closed curve, squares and triangles are not distinguished from circles, and the drawing is of two or more or less intersecting lines. While there is no distinction as yet between straight-sided and curved figures, depending on the complexity of the figures, there is correct copying of the topological properties.

Stage II (4-6 years) marks the beginnings of the differentiation of what Piaget designates as Euclidean shapes. Thus, a square is distinct from a triangle, a circle from an ellipse. Two types of crosses are distinguished which marks the discovery of oblique lines. And finally, the rhombus, (as distinct from the square), is reproduced correctly.

At Stage III (6 to 7 years) the child's idea of shapes is at the operational level. Children draw figures quickly and correctly and these constructions reflect anticipation through mental images (advanced organization) in terms of potential measurements, coordinations, etc.

Development of Topological Space

The study of drawings has shown that the simplest topological relationships such as proximity and separation also are the first to emerge in the course of psychological development. This order of appearance is also maintained when space is treated axiomatically by geometricians. In the case of a linear series, the relationship of proximity subsisting between separate elements A, B, C, is sufficient to provide a basis for the relation of order. And this may be perceived intuitively at an equally early stage of development. The notion of order or sequence is thus a third basic topological relationship.

The relation of order exemplified by three elements arranged in a series ABC also entails a specific relationship expressed by the word 'between.' Thus B is 'between' A and C, and at the same time 'between' C and A. This relationship, whose invariance remains a mystery to children who have not yet learned to reverse a series, evolves concurrently with the notion of order itself. The relation between is one particular instance of the more general relationships of 'surrounding'. These are, of course, elementary spatial relationships, just as much as proximity, separation, or order. Indeed, as regards the construction of space, they are even more important, since it is most probably these relationships which lead the child by the most direct route to differentiate and build up the three initial topological dimensions.

If the location of a point between two others designates a one-dimensional surrounding (i.e., a line), and the location of a point inside or outside a closed figure designates a two-dimensional surrounding (i.e., a surface), then the relationship of a point, whether inside or outside a closed box, designates a three-dimensional surrounding (i.e., 3-dimensional space).

It appears that in the case of 'surrounding', there is one area in which perceptual relationships have not yet been developed and hence is ideally suited for studying the main features of representation. Knots have the added advantage of having been the subject of extremely detailed geometrical analysis. From the standpoint of mental development, the knot is something which the child learns to form at an early age and it is therefore eminently suited to psychogenetic investigation. The task consists of asking the child to (1) reproduce an 'overhand' knot (an ordinary single-looped knot), a circle, a figure of eight, or a pseudo-knot (homeomorphic with the circle when the ends of the string are joined); (2) to compare left vs. right overhand knots, taut vs. slack knots, and an overhand knot vs. a circle, a figure of eight, or a pseudo- knot (3) to predict the shape of the knot following certain transformations.

Stage I (up to 4) can be divided into two substages. During Substage IA the children could not copy knots. Either one end of the string is wound around the other without inserting either end in the loop or one end is inserted in a half-loop without superimposing it. In neither case is there the necessary 'surrounding' and, consequently, no knot. During Substage IB the children learn how to copy the knots but are unable to follow the various sections of a slack knot with one finger, nor were they able to distinguish true from pseudo-knots.

At Substage IIA the child perceives the identity between a pair of taut knots, or between a pair of loose knots, but this identity is lost if one of the knots is tightened (or slackened) even though each of the pair is homeomorphic to the other.

A parallelism between the concept of number and the concept of space can be found at this transitional period. At about 5 or 6 years of age, a child can establish intuitively the correspondence between a number of objects and

125

separate but equal number of other objects, but only when the arrangement of objects produces a similar visual pattern (such as two straight rows). As soon as the intervals between consecutive items in one of the rows is altered the equivalence of number is no longer recognized. Similarly, in the experiment reported above, the child at 5 or 6 can recognize either tight or slack knots when compared with visually identical models, but is unwilling to grant their equivalence without the perceptual equivalence. At Stage IIB the correspondence between the taut and slack knot is established through motor anticipation. For example, the child might say: "If I pull, I'll get the one before" (Piaget, 1948). In other words, the child at this stage can imagine the knot in terms of actual transformational motor activities, rather than perceptually static patterns.

Finally, at Stage III (8-10 years), the actions become internalized and completely reversible in nature. Whatever happens to the perceptual patterns, the relationship of the 'surrounding' remains unchanged.

Development of Projective Space

As has been pointed out earlier, topological space only furnishes the basis for that type of analysis which operates from the standpoint of each figural object considered in isolation. In projective space, however, the object is considered in relation to the viewpoint of the subject (perspective) or that of other objects (projective). Thus, projective relationships presume the inter-coordination of objects separated in space.

Piaget's reports of his investigations concerned with projective lines and perspective, attempt to show that the precondition for forming projective straight lines is a progressive discrimination and coordination of different viewpoints. Next he takes up projection of shadows and provides a demonstration that projection of shadows is understood at the same level of development as perspective transformations relating to the same object. The study on "viewpoints" (of a group of mountains) the problem of the overall coordination of perspectives, such as arises when the observer moves around and about a number of interrelated objects, is analyzed. The mutual implications of projective and Euclidean space notions is then studied with the rotation and development of surfaces and the distinction between the perceptual and conceptual space is considered. This view will be specifically concerned only with the problems of construction of a straight line and coordination of perspectives. The former involves perspective of a single object, while the latter with perspective vis-a-vis a group of objects.

The concept of straight line results from the child's first attempts to relate objects spatially in a system of projective viewpoints or coordinates. Strictly speaking, the topological idea of 'a line' does not include the 'straight line' at all. To transform an ordinary line (the only kind of line recognized by topology) into a straight line requires the introduction, either of a system of viewpoints such as the elements of a line masking each other to form a perspective, or else a system of displacements, distances and measurements.

The task of constructing a straight line will illustrate clearly the critical difference between perceptual and conceptual space. The task consists of asking the child to use match-sticks to form a straight line parallel to the edge of a square table, lying at some angle to the two adjacent sides of the square table, or across a section of a round table.

The results show that at Stage I children can recognize a straight line and distinguish it from a curve, but they are unable to construct such a line parallel to the edge of a table, except when allowed to do so where an existing model or edge of the table is spatially very close. Otherwise, the performance reflects the formation of a topological line with successive elements very close together and curved in various ways. When the line to be imagined and constructed conflicts with perceived straight or curved lines lying adjacent, such as on the 'ground' offered by the table top, the child is no longer able to form a straight line. In this case, the line no longer consists of merely imitating a past or present perception, but entails creating new relationships within an existing pattern distinct from those sought after. Such an achievement requires either a projective operation based on the action of 'taking aim', or else a Euclidean operation based on change of position.

At Substage IIA (4-6 years), the child arranges the matches parallel with the edge of the table, and can even arrange them against a neutral ground. But, he is unable to resist the influence of an edge (of the table) if the required line is no longer parallel to it. Two distinct types of spatial concepts need to be recognized at this point. The first, intuitive, is no more than an internalized imitation (a mental image) of previously perceived events. Consequently, it can be either favored or discouraged by current perceptual configurations. The second, not evolved as yet, is based on operations, and therefore is freed from the influence of such configurations.

At Substage IIB (6-7 years) the child is able to free himself from the influence of the surrounding perceptual configuration and to form lines independent of the edges of the table. At Stage III (8-11 years) the child discovers a new technique of visual alignment, i.e., the operations of 'sighting' or 'taking aim'. The child discovers the projective straight line, then, when the fact that two points X and Y can be related to the observer O through the agency of the line OXY is grasped. The conceptual straight line thus differs from the perceptual straight line (and from the topological line) by virtue of the awareness of the part played by different points of view. To join X and Y together in a direct line, the child must, at the same time separate them from the perceptual ground, join X and Y either by means of a movement or else by visual inspection. The second procedure can only be carried out by discriminating between different points of view, and it is choice of the point of view OXY that enables the child to correct his alighment.

The perspective of a group of objects as viewed by an observer from different positions, or alternatively by a number of observers, is examined with two aims in mind. First, to study the construction of a global system linking together a number of different perspectives, and secondly, to examine the relationships which the child establishes between his own viewpoint and those of other observers.

The task used by Piaget consists of three clearly distinguishable model mountains on a pasteboard. The child is asked to imagine, or to identify from pictures, the various views (or 'snapshots') of the group of mountains when it is seen from different positions.

The results show that throughout Stage II (5-7 years) the child does not distinguish between his viewpoint and that of other observers. At Substage IIA, performance is confined to reproduction of his own point of view.[6] In Substage IIB the child makes some attempt to distinguish between different viewpoints, but usually lapses into the egocentric constructions of Substage IIA. The child does not yet think in terms of 'groupings' of projective relations and correspondences or the invariance of the correspondences amid the endless transformations of the projective relationships.

At Stage III (7-8 to 11-12 years) the child shows a progressive discrimination and coordination of perspectives. At Substage IIIA certain relationships are varied with changes in the position of the observer, but there is still no comprehensive coordination of viewpoints. In most cases, the before-behind relationship can be reversed, but the left-right relationship retains its rigidity. At Substage IIIB the child achieves a complete relativity of perspectives as demonstrated by the discovery that : (1) to each position of the observer there corresponds a particular set of left-right, before-behind relations between the objects constituting the group of mountains, i.e., the point to point correspondence between position and perspective; and (2) between each perspective viewpoint valid for a given position of the observer and each of the others, there also is a correspondence expressed by specific changes of left-right, before-behind relations. It is this correspondence between all possible points of view which constitutes coordination of perspectives.

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[6] It is the egocentric illusion which prevents these children from reversing left-right/before-behind relations and thereby rotating the perspectives along with the changing viewpoints, a continuation of the illusion which is responsible for absence of shape constancy in young babies' perception.

From the point of view of both mathematical construction and psychological development, projective and Euclidean space are closely related and both derive from topological space. In addition, projective andEuclidean space are related in another way; i.e., it is possible to construct a series of transitional stages between projective and Euclidean space by considering affinities and similarities. Affinities may be defined mathematically as projective correspondences conserving parallelisms; similarities as affinities conserving angles; and Euclidean displacements as similarities conserving distances.

The conservation of parallels can be demonstrated in the study of reactions in a very simple case of 'affinitive' transformations, namely,the increase and decrease in the width of the rhombuses in a set of "Lazy Tongs" (Fig. 1)

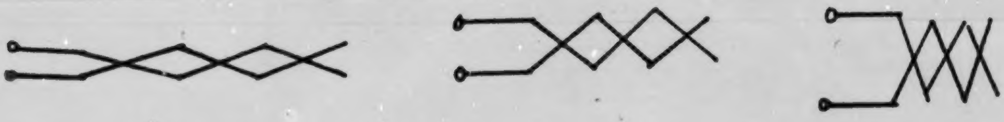

Figure 1. The transformations of rhombuses seen in "Lazy Tongs"

The task consists of asking the child to predict and draw what will happen when the handles are opened or closed.

The results show that the child at Substage IIA (4-5½ years) is unable to anticipate any transformation when the apparatus is stationery. If he sees the beginning of a change, he can imagine the continuation of it, but only in the form of endless enlargement of the 'windows.' At Substage IIB (5 ½ - 7 years) the child foresees that the rhombuses will grow longer and he is also prepared to admit that they will eventually get smaller again. But this prediction remains global and inadequate, for the transformations appear to him not as a continuous (quantitative) series but as abrupt, disjointed changes (qualitative).

At Stage III (7-11 years) all the transformations are predicted correctly and the necessary parallelism of opposite sides of the rhombus is recognized; i.e., the parallelism of the sides is operationally lined with the transformation of the figure as a whole and is no longer simply perceived or imagined intuitively.

The next task is to examine the discovery of the similarity of angles, such as in triangles or rectangles, in order to analyze similarities. To do this, children were asked to draw, or compare, pairs of similar or dissimilar triangles.

The results show that at Substage IIA (4-6 years) the child faced with the task of drawing a triangle larger than the model one, is content with producing any sort of triangle, i.e., all triangles are treated as the same in contrast to non-triangular figures. During Substage IIB (6-7 years) the enlargements begin to take account of parallelism between the pairs of sides, but only in a few special cases, such as when the enlargement is relatively small and the parallelism is apparent.

Stage III (7-11 years) marks the appearance of operations facilitating general comparison of parallels, angles and simple dimensional relations, such as the ratio of 1:2. At this level, the child is able to draw and compare similar triangles, but not so with rectangles. The latter requires some knowledge of dimensional proportions whereas recognition of similarity of triangles depends only on elementary qualitative operations. As Stage IV, the child attains a concept of proportionality for all dimensional relations.

Development of Euclidean Space

Elementary topological relations are concerned with the object as a thing-in-itself and/or with various features taken successively. Projective concepts, in contrast, imply a comprehensive linking together of separate objects into a single system based on coordination of different viewpoints. Concurrently with the development of an organized complex of viewpoints, coordination of objects as such emerges. Ultimately this latter development provides for the transition to Euclidean space, with the concepts of parallels, angles and proportion providing the link between the two systems. Such a coordination of objects requires conservation of distance together with some notion of 'displacement' (or congruent transformation of spatial figures) which culminate in the construction of systems of reference or coordinates.

At the outset, the coordinates of Euclidean space appear to be no more than a network embracing all objects and consists of relations of order applied simultaneously to all three dimensions. Within this network each object is linked simultaneously with the rest in three directions; left-right, above-below and before-behind, along straight lines parallel to each other along one dimension and intersecting those belonging to the other two dimensions at right angles.

A reference frame proper, however, is not simply a network composed of relations of order between the various objects. It applies equally to positions and enables the relations between objects to be maintained as invariant and independent of potential displacement of the objects. Thus, a frame of reference constitutes a Euclidean space after the fashion of a 'container' and is relatively independent of the mobile objects contained therein; just as projective co-ordination includes the totality of potential viewpoints.

The simplest and most natural reference frame available to the child is probably that provided by the physical world in the shape of vertical and horizontal axes. On the empirical level, the horizontal is given by the plane on which everyday objects rest, the earth itself (where flat), or the artificial planes of floors, terraces, and so on. Another important factor is the surface of a liquid; e.g., surface of a lake or level of water in a glass. Walls of rooms and houses, by posts, chimney stacks, trees, etc. provide experience with verticality. The study of the construction of horizontal and vertical axes provides a suitable way to understand the construction of coordinates in Euclidean space.

The task for studying horizontality consists of two narrow-necked bottles, one with straight, parallel sides and the other with rounded sides. Each is partially filled with colored water and the children are asked to "guess" the position of the water when the bottle is tilted. The study of the vertical involves a small cork with a matchstick placed vertically in it and floating on the surface of the water in the jars. The child is asked to draw the position of the "mast" of this "ship" at different inclinations of the jar. In addition, the child is shown a mountain of sand, etc. and asked to plant posts "nice and straight" on the summit, on the ground nearby, and on the slopes of the mountain.

The results show that at Stage I (up to 4-5 years), the child is unable to represent either the water or the mountain as a plane surface. At Stage IIA the child realizes the water as a plane surface, but always parallel to the base of the bottle even when the bottle is tilted. It is surprising, but children at this level not only fail to note that water level is always horizontal in their everyday observations but also do not use the results of the experiment when it is performed (i.e., with complete perceptual information available!)

During Stage IIB, the child no longer draws the water level parallel to the base of the jar when the jar is tilted, but he still fails to coordinate his predictions with any fixed reference system outside the jar (i.e., with the table or the stand). Also, children at this level are usually able to stand the posts upright when planting them in the sides of the sand hill, but continue to draw them perpendicular to the incline.

At Stage III (7-8 to 9 years), the child is able to predict horizontal and vertical in all cases which requires a system of coordination or series of comparisons between objects in different positions and orientations. These comparisons are made by linking the various objects together in a system where stationary objects (the table, etc.) serve as reference points for mobile ones (the liquid surface).

The conclusion to be drawn is that Euclidean relations, completed by the construction of reference frames, are essentially relations established between numbers of objects and serve to locate them within an organized whole. Thus, horizontal-vertical axes are constructed at the same time as perspectives and are coordinated since these latter also constitute overall systems linking together objects or patterns. But projective space is in essence a coordination both of viewpoints, actual or virtual, and of the figures considered in relation to these viewpoints. Coordinates, on the other hand, link together objects, such as, in their objective positions, displacements, and relative distances. The age of 9 (approximately) is midway in the period during which concrete operations first take shape and thus mark a decisive turning point in the development of spatial concepts; i.e., completion of the framework appropriate to both Euclidean and projective systems.

Conclusions

Piaget maintains that the evolution of spatial relations proceeds at the two distinct levels of perception and operational (conceptual). Several years of experience separate the full development of perceptual and conceptual construction of space, but similar paths of development (topological, followed by projective and Euclidean) are observed.

Topological relations (proximity, separation, order, enclosure and continuity) are the most elementary spatial relationships both from the genetic-psychological and mathematical point of view. The topological relations, with which the child begins to construct his concept of space, are transformed concurrently into projective and Euclidean concepts. Projective space introduces a 'point of view' and Euclidean space introduces 'distance' and 'measurements,' into topological space. The first of these, embracing perspective, section, projections, and plane rotations, results from the coordination of viewpoints, while the second derives from the conservation of straight lines, parallels, angles, and lastly, general coordinate systems.

In psychological terms, topological space relations consist of such elements as "A is near, beside, far from, in(side), out(side) B," or "A is between B and C." Projective space introduces relations such as "A is before/behind, above/below, or to the right of/to the left of B" and Euclidean concepts add a quantitative aspect to these projective relationships; i.e., Euclidean relations consist of the additional concepts such as "How Far is A before/behind, above/below, or to the right/to the left of B?"

Elementary topological relationships subsist between neighboring parts of a single object, or between an object and its immediate environment. Such a space is merely a continuous collection of elements which may be expanded or contracted; neither straight lines, distances, nor angles are conserved. Consequently, topological concepts do not lead to the construction of a stable system of figures, nor to fixed relations between such figures. Topologically, each continuous domain constitutes a space and there is thus no universal space operating as a frame and enabling objects or figures to be located relative to one another. There are, for the child, as many spaces as there are objects or distinct patterns, the intervals between more distant elements either belonging to the elements themselves or not being spatial at all. In this connection, projective concepts perform a vital role in bringing about a global coordination of space.

Projective concepts take account, not only of internal topological relationships, but also of the shapes of figures, their relative positions and apparent distances, though always in relation to a specific point of view. Unlike the coordinate system implied in Euclidean space, a projective system does not conserve distances and dimensions, but does conserve the relative positions of parts of figures or of figures relative to one another and the whole in relation to the plane corresponding to the observer's visual field. From the psychological standpoint, the essential feature here is the inclusion of the observer (or a 'point-of-view') in relation to which the figures are projected. Elementary projective concepts are therefore based on the same operations as are topological concepts, but with the addition of a 'viewpoint'. The linking this 'viewpoint' with operations of order is basic for the construction of the projective straight line. A straight line is a series of points so arranged that from the 'end-on' viewpoint they are in alignment and are reduced projectively to a single point. Similarly, the notion of spatial dimensions can be defined more clearly in terms of certain sets of conditions specified by a given viewpoint. Topologically, the first dimension corresponds to a linear series, the second to the notion of inside and outside a closed linear boundary, and the third to the notion of inside and outside a closed two-dimensional boundary (surface). The addition of a perspective viewpoint to which the figures are related permits these same relationships to embrace relative orientation as expressed by 'on the left or on the right', 'above or below', and 'before or behind'.

Intermediate between the projective and Euclidean relationships, there arise certain relationships (affine and similarity) which children begin to understand at about the same time as they master the projective relationships. Projective relationships conserve neither parallels, angles, nor distances. Affine relationships, on the other hand, conserve parallels while angles and distances continue to vary. In the case of similarities, however, the figure retains its shape unchanged (straight or curved lines, parallels and angles) but changes in size according to relations of proportionality. Finally, Eudlidean relationships add the notions of distance and measurement.

To conclude, topological and projective geometry are concerned with qualitative properties of space, whereas Euclidean geometry introduces quantitative properties to space.

Child's Conception of Geometry

The techniques Piaget (Piaget, Inhelder and Szeminska, 1960) used to study conservation and measurement of length, area, and volume will first be described briefly. Then, some general findings will be presented with the processes assumed to be involved in the child's construction of metric properties of space considered last.

Methods

The results obtained from studies of children's spontaneous measurements and comparisons of two straight lines are described in some detail so as to clarify the general context in which measurement and conservation behaviors develop.

Spontaneous measurement was studied by showing a tower made of twelve blocks of cubes and parallelepipeds. The tower was 80 cm in height and the child was asked to build a similar tower on another table. The instructions deliberately avoided mention of measurement; the experimenter used phrases such as: "You make a tower the same height as mine."

For conservation of length the child first was shown two straight sticks identical in length and with the ends aligned. One of the sticks was then moved forward 1 or 2 cm, and the child asked which of the two was longer or whether they were the same length. In a second task, twelve to sixteen matches were arranged in two parallel rows and 1-2 cm apart. One of the rows was then modified by placing the matches at an angle. The question to be answered was always whether the two lines were still the same length. Finally two strips of paper each 30 cm long and about 1 cm wide were placed before the child and he was asked to assure himself the two strips were identical in length. One of the strips was cut, first in two parts, then in several, and parts arranged in an arc, etc. The same questions as those in the first task were asked.

The technique for studying measurement of length was a direct extension of that used in the conservation of length task. The child was asked to judge between strips of paper in a variety of linear arrangements, involving right-angles, acute angles, etc., but these were now pasted on cardboard sheets. After his replies ("equal" or "one is longer") he was shown several movable strips and asked to verify his judgment: "Have a look with this and see if you're right. Try and measure," etc. Finally, he was given short strips of cardboard 3 cm, 6 cm, (these lengths corresponding with those of segments on the mounted strips) to aid in verifying his judgment. The experimenter also demonstrated by applying the 3 cm card two or three times along the mounted strips, beginning with the point of origin, and explained "a little man is walking along a road and these are the successive 'steps' he takes as he walks." In this latter case, the child was asked to finish the task as he was shown.

The conservation of area tasks were composed of several separate sections which permitted a modification of the arrangement of parts to test whether or not the child considered the whole to remain constant. For example, two cardboard rectangles, each made up of 6 squares, might be used. The twelve such squares were all equal and each of the rectangles was two squares wide and three high. After constructing the rectangles, the experimenter transferred the top right-hand square on one to the bottom right-hand corner, which yielded a pyramid of three squares in the bottom row, two in the second, and one at the top. The child was asked whether this figure had the same area as the other rectangle which was left intact. In a second task the child was shown two rectangles, recognized as congruent, from which the experimenter cut off a portion of four corners, putting them against the sides to produce an irregular polygon, etc. (any congruent figures can be used instead of rectangles if desired). The questions were always: "Are these the same size?" "Is there the same amount of room?" etc.

Measurement of areas was studied by using two tasks: (1) Measurement by superposition involved test objects consisting of a large right triangle (A), an irregular figure (B), and triangles (squares cut diagonally in half). There were sufficient smaller shapes to cover the whole of B and more than enough to cover A. The child was asked to use the smaller shapes to cover the large figures. (2) For measurement by unit iteration the child was shown a number of shapes equal in area but differing markedly in shape. One (A) was a square that could be composed from the nine smaller squares and the others (B and C) were irregular figures made up of the same number of small squares. The child was given a cardboard square representing one unit together with a pencil and was free to examine the material, and to draw on it. If he did not know what to do, the experimenter demonstrated how to use the unit-square to cover the large square if necessary. When finished with A, B, and C, the child was given two more shapes that were more heterogeneous and not equal (D and E). The child was offered a choice of three counters to use for measuring: a square, which was a quarter of D, a rectangle double the squares (so that two would fit into D), and a triangle equal to a square cut diagonally in half. The child was asked to use the small squares or triangles to fit into the large figures.

Conservation and measurement of volume was observed by showing the child a block measuring 4 cm in height with 3 cm x 3 cm base (volume - 36 cm). The block was presented as "an old house" built on an island, (a square cardboard, 3 cm x 3 cm pasted on a sheet of corrugated card). The house is "threatened," so the inhabitants decide to build another which is to have exactly as much room. The child is shown these other islands which are also pieces of card but which differ from the first in size or shape or in both, their measurements being 2 x 2 cm, 2 x 3 cm, 1 x 2 cm, 1 x 1 cm and 3 x 4 cm. The problem consists in reproducing the volume of the first block while altering its form to comply with the new base. The equal volume must be built from wooden cubes of 1 cm^3 (the original block is solid). Equality of volume was expressed as: "as much room," with further explanation as necessary.

Results

The results are presented here in three sections: first, results from the study of children's spontaneous measurements; second, the results from the study of the comparison of two straight lines; and third, a summary of the results of studies of conservation and measurement of length, area, and volume.

Spontaneous Measurement. The responses made by children in the study of spontaneous measurement are summarized in Table 2.

Table 2

Levels of Development in Childrens' Spontaneous Measurement

Stage 1	(4-4½ years)	Visual transfer
Stage II	(4½ - 7 years)	IIA: Manual transfer IIB: Body transfer
Stage III	(7 - 8½ years)	IIIA: Transitive congruence IIIB: Unit iteration

At Stage I, visual transfer is the only basis of comparison between the two objects (e.g., towers), comparison of the heights of two towers is made entirely by moving the line of vision - the subject makes no effort to move one of the towers closer to the other.

Substage IIA is characterized by visual transfer being supplemented by manual transfer. The towers to be compared are now brought together so that an appraisal of "neighboring" objects is made. In Substage IIB children use their bodies, e.g., the span of hands or arms to "transfer" the height of the towers from one to another (body transfer). Such behavior is considered the beginning of the use of a middle term, but the transitivity involved here is still intuitive.

During Stage III transitivity in the operational sense is understood. The smaller as well as the larger object (term) is used as the middle term. For example, the length required is noted; if the "rule" is too short, they stop it and go back and forth between the towers. Eventually, any object available is used as a common measuring rule. Such an object is stopped as often as necessary, which is the equivalent of assigning a "unit-value" to a given length. This operation of unit iteration, marks the appearance of a metric system.

Comparisons of Length. The results of comparing two straight lines are summarized in Table 3.

Table 3

Levels of Development in Children's Comparisons of Two Straight Lines

Stage I & IIA	Non conservation
Stage IIB	Intermediate responses
Stage III	Conservation

Stage I & IIA behavior is characterized by no conservation of length; i.e., judgments depend exclusively on the perceptual characteristics of the setting. Thus, when the child visibly focuses on the leading extremity of a moving stick, that stick will be judged longer and the progressive changes at the other end of the stick is ignored.

The beginning of Stage IIB is observed when intermediate responses occur; i.e., the child responses oscillates between non-conservation and conservation. For example, the child concentrates first on one end of the pair of sticks, and judges the top stick to be longer (because it projects beyond the other), but the next moment he focuses on the other end-point where the lower stick projects beyond the top one and he now decides that the lower stick is longer.

Conservation of length defines the Stage III level and results from the operation of compensation. When the sticks are being staggered the child often responds: "The sticks are still the same length; ther's a little space here (difference between the leading extremities) and there's the same little space there (difference between the trailing extremities)."

Summary

The development of conservation and measurement in length, area, and volume follows similar courses as those observed in the studies of straight line comparisons and spontaneous measurement. The only differences lie in: (1) Content, i.e., the number of dimensions involved; (2) The measurement of volume is relatively inadequate at level IIIB as compared to that length and area at the same level. One cannot apply a unit-volume over the total volume to be measured in most cases as is possible for length and area; i.e., often many elements are hidden from view; and (3) Calculation of length based on linear units appear complete at level IIIB, but not so for area and volume. Again, the child can perform unit iteration at IIIB level but "multiplication" (e.g., for a rectangle, 2 cm x 3 cm = 3 cm) operations are not realized until Stage IV.

Responses of the children in the experiments of conservation and measurement of length, area, and volume are summarized in Table 4.

Table 4

Summary of children's Responses in Conservation and Measurement

	Conservation	Measurement
Stage I & IIA	Non conservation	Perceptual comparisons
Stage IIB	Intermediate Responses	Intuitive transitivity*
Stage IIIA	Conservation	Operational transitivity*
Stage IIIB		Metric measurement: unit iteration***
Stage IV**		Metric measurement: mathematical multiplication***

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*Transitivity is intuitive when the transferring of a middle term in comparison is limited only to some certain favorable situations (e.g., when objects to be compared are neighboring each other), and the comparison is still not free from perceptual influence and thus is still only approximate. An operational transitivity on the other hand, is not restricted by the situational factors; it depends solely on logical inference, thus, if A=B, and B=C, then A=C.

**Stage IV applies only to the development in the measurement of area and volume.

***The operation of unit iteration is not equivalent to mathematical multiplication. The former consists in measuring the size of an object by moving a unit measure of the same dimensions stepwise over the total object (i.e., using unit-length to measure length, unit-area to measure area, unit-volume to measure volume). The latter allows one to measure an object using units of fewer dimensions. Thus one can measure area using linear units (e.g., 2 cm x 2 cm = 4 cm^2); or one can measure volume using linear units (2 cm x 2 cm x 2 cm = 8 cm^3), or linear and two-dimensional units (e.g., 4 cm^2 x 2 cm = 8 cm^3).

Discussion

To measure requires some "thing" as a base unit and transposing of that unit in a systematic way to the whole of the object to be measured. Underlying the Piagetian concept of measurement are the concepts of "conservation of size," "subdivision," "change of position," and a "coordinate system." The measurement of space, thus, is not possible without the establishment of conservation of size, and of the coordination between change of position and subdivision.

The achievement of conservation of size depends on recognition of the distinction between empty space as "container" and solid movable objects as "contained." If, for example, two straight sticks of equal length are first laid end to end and then slightly staggered in relation to one another, Stage I and II subjects deny that the lengths are equal, but at Stage III they are convinced of the equality because now it is "recognized" that newly occupied "sites" compensate for places left empty by the change in position. This awareness of compensation is based on the discovery that properties of length, area, or volume remain invariant when position is changed. But, the discovery of these invariants in turn depends on knowledge that when an object undergoes a slight change of position the space left unoccupied by the change is exactly equivalent to the newly occupied space. While the argument is circular - since compensation between empty spaces and newly occupied spaces depends on the invariance of area and volume despite change in position, and the latter depends on the awareness of compensation - it is this reasoning which enables the child to recognize the conservation of size when objects undergo changes of positions.

Conservation of size is but a precondition for measurement. Measurement also depends on the coordination of subdivision and change of position. The operations of subdivision and change of position and their relations to measurement are as follows:

Operations of subdivision. Let us suppose a straight line C with a given length. By the operations of subdivision and composition the line may be broken into a number of contiguous parts and these in turn can be re-united to compose the original whole. Such parts may be represented by a length A which, together with another A' to which it is contiguous, yields a more inclusive part B ($= A + A'$), while B in turn is completed by a final part B', so that together they yield the whole line ($B + B' = C$). A child at Stage IIIA understands composition and can therefore deduce the following relations: $A + A' + B' = C$; $B - A' = A$; $C - B = B'$, etc. Here the quantification is derived only from part-whole relations and not from relations between one part and another: the subject is aware that $A<C$; $A'<C$; $B<C$; $B'<C$; $A<B$, etc., without needing to know the precise lengths of A, A' and B'. Between these elementary parts, A, A', and B', there can be no relation other than that of qualitative equivalence which is derived from their common membership of C.

Operations of Change of Position. In the linear series above, the initial order of its three elementary parts was AA'B'. Any change in their relative position is simply a matter of altering that order to A'AB or AB'A etc. Similarly, to change the position of the total line C amounts to an alteration in the order of C relative to a series of reference elements. On the other hand, any change in the order of its own parts will not affect the total length (C) because $A' + A + B' = B' + A + A' = A + A' + B = C$. Likewise, the forward movement of the line C does not affect its length; i.e., the compensation is between newly filled spaces and vacated spaces. But, compensation also can be expressed in the language of qualitative subdivision: by moving forward, the stick C is increased by a new element C' at its forward end; at the same time it loses a part C'' which it leaves behind. But since change of position is a change in the order of things, the new part C' and the old part C'' are regarded as equivalent, i.e., $C' = C''$, and the child argues $C + C' - C'' = C$ just as $C + C'' - C' = C$.

Thus, both subdivision and change of position are qualitative in character, and each taken alone is not sufficient to give rise to measurement. This fact is clearly seen in the study of children's spontaneous measurement. A change of position without subdivision evokes intuitive judgments based on movement. The most elementary form of this response is for the subject to run his finger along the two lines and make a motor comparison. A slight advance is noted when the child transfers a span or the width of two or three fingers. However, all these response types are limited because subdivision is approximate; there are no definite "marks" for guidence. Subdivision without change of position also cannot result in measurement. The most elementary form is exemplified by responses of a child when asked to judge the length of two outlines. He lays a strip of paper along one part only of a single outline. Later, he lays another strip along part of the second outline and passes judgment on the relations between the two, without comparing the two measuring strips. What he calls measuring is simply comparing two outlines by splitting each one separately into sections,without transferring these sections from one outline to the other or comparing them with each other. Thus, when subdivision without change of position occurs, two objects cannot be compared in terms of metric unit, and there is no true measurement.

Measurement begins when one part (A) belonging to a whole (C) is compared with the remaining parts of the same whole by change of position (either its own or that of a common measure, used transitively) so that A (or its equivalent) is superposed on these other parts. This implies that subdivision and change of position are fused into one single operation and no longer simply complementary. The operations alluded to above involve the alternate use of subdivision and change of position and not the two together. Thus, subdivision antecedes change of position and is not its consequence; but change of position itself also is quite independent of subdivision. This fact is illustrated in the conservation of size. There is always an initial subdivision of the whole into parts and the relative position of parts is then changed so the various parts take one another's positions. The operations do not involve any direct or indirect comparison between the several parts. But when one part (A) is applied to the remainder of the whole (C), the subdivision is not independently given, it is generated by the change of position effected either by A itself or by its transitive equivalent. Thus, A is moved stepwise along C-A, giving first A=A' (so that B=2A), then A=A' = B' (so that C=3A). In this case the subdivision cannot be dissociated from the corresponding changes of position. The subdivision depends wholly on change of position, but the reverse is also true. It is this synthesis of qualitative operations which gives rise to unit iteration and so constitutes measurement. By applying one section over and over until the whole has been covered completely, the whole is effectively reduced to a multiple of that section and the section becomes a unit. Since the unit can be subdivided in turn, using one of its fractional elements as a sub-unit, it follows that any size whatever can be compared with any other by means of whole and fractional units.

The notion of unit iteration to this point, however, is not yet completely developed in the metric notion of space. Areas are measured only in terms of units which are themselves areas, and volumes in terms of unit-volumes. It is not until later that the child learns to calculate areas and volumes by linear units. This ability to apply mathematical multiplication to the measurement of space evolves, according to Piaget, from the realization of the continuity of space. Thus, surface can be reduced to line when it is thought of as an infinite series of lines; volume can be reduced to surface when it is thought of as an infinite series of surfaces. It is with this achievement (mathematical multiplication) involving area and volume that the child has reached the final phase in the reconstruction of Euclidean space.

To summarize, geometry is the science of space. The child's notion of space changes with development. At first, the child only is able to conceive of space in terms of such relationship as neighborhood, order, betweeness and closure. Later, he learns to construct space by a 'point of view' of the observer(s), and to describe space in terms of left-right, before-behind, and above-below. At the final stage, the child can conserve distance, and with the aid of a coordinate system, begins to conceive space in metric terms.

The three stages in the evolution of the child's notions of space correspond to spatial relationships constituting three branches of geometry. The conceptualizations that appear first in psychological development are the ones constituting the topological space; those appearing next constitute projective space; and those appearing last in psychological development are ones that constitute Euclidean space. The evolution of those notions of spatial relationships constituting topological and projective spaces has been treated in The Child's Conception of Space (Piaget, 1948). The development of the child's understanding of Euclidean space was left to a separate volume because it involves the complex notions of distance and measurement. The Child's Conception of Geometry (Piaget, Inhelder and Szemiska, 1960) continues the work by treating these problems. In the latter volume, it was argued that conservation of distance is based on the distinction between movable objects and fixed sites as reference points. Measurement of space presupposes the notion of the conservation of distance and actually begins as the operations of subdivision and change of position are fused into a single operation.

Final Thoughts and Directions for Future Inquiry

A theory as comprehensive as Piaget's quite naturally draws much attention. Replications are attempted, discrepancies are found, and research based on the theory tends to, initially, be multi-directional. It would be surprising, indeed, if no findings contradictory to the theory were reported. What is surprising in this case, is that so much of Piaget's findings have withstood the empirical onslaught. And, Piaget's (1970) more recent distinction between figurative and operative thought (Smock, 1969) gives the theory more flexibility for dealing with a broader range of psychological problems, including "real life activity." Youniss and Dennison (1971) have confirmed some of the implications of the figurative and operative distinctions - showing that the two processes are complementary within developmental levels but with operative thought dominating the figurative processes. Also, Piaget's theoretical structure too often has been considered without the role it gives to experience, which varies from one individual to another; the critical point is that modes of dealing with experience is similar among persons.

Piaget's biological tradition has often been contrasted with the psychometric tradition in the U. S. but the two approaches may not be so far apart. Laurendeau and Pinard's (1970) extensive study of five Piagetian tasks, which confirmed many of Piaget's findings using a great many subjects, made good use of scalogram analysis. The technique is gaining in popularity, since it supposedly answers the question of whether a set of behaviors occur in an orderly (developmental) sequence. Laureudeau and Pinard's research also sets an important precedent; that is to use the same children for many tasks within the same study. The analysis of inter-relations among task performance of the same children is necessary to describe the common underlying cognitive structure. For example, Kaufman (1971) examined the factor structure of some of Piaget's and Gesell's tests, and the Lorge-Thorndike tests in 5-6 year old children. The Piagetian tasks reflected three factors, corresponding to the operation of number, classes, and relations. Bart (1971) reports a general formal operational factor, and one related to task context, whereas Berzonsky (1971) found five separate factors (in 30 variables with first-grade children). Shantz (1966) did not use factor analysis, but did study the interrelationship among tests of class multiplication of asymmetric logical relations, and spatial multiplication in concrete operational children. The tasks were found to intercorrelate moderately, but exhibited a variety of more specific complex relationships.

A point often neglected is that a moderate correlation between scores on tests of multiplicative abilities, for example, is not to say that the relationship between multiplicative abilities is moderate. The latter statement could be made if all relevant variables are considered, but it is difficult to know when this has been achieved since adequate data is still not available. Researchers, understandably, wish to develop their own tasks. Then, contradictory results lead to questions of validity of the instruments. Even the validity of the Piagetian tasks (since primacy does not imply validity) is not known. It does seem, however, that too many new tests are developed prior to a thorough understanding of the theoretical imperatives for tasks that might provide a test of the theory. There is the nagging suspicion that some tests are developed with a hopeful eye toward being able to test a child and, with certainty, place him in Stage X. The dangers of assessing a child's cognitive structure(s) on the basis of a few "tests," especially using standardized psychomatic methodology, was pointed out early by Inhelder, Smock, Bovet and Sinclair (1966).

One further thought. Gagné (1967) would leave one devoid of any hope for Piaget's theory. "I believe that many of the principles mentioned by Piaget, such . . . as reversibility . . . are abstractions . . . obviously in Piaget's mind. But they are not in the child's mind" (pp.188). Surely what Gagné means is that an (external) observer attaches a label to mental operations. It really is unlikely that Piaget thinks the child goes around saying, "I am practicing reversibility." When it comes to the brass tacks of a counter theory, Gagné can only offer that specific "euthenics" which are learned are generalized by combining with other "euthenics," "by means of a little understood, but nevertheless dependable mechanism of learning transfer." Our only comment is that this phantom mechanism also has gone unnamed by all children I know.

What one does when two theoretical positions are in conflict is rarely a matter of empirical or logical compulsion. The learning and developmental approaches often speak of the same phenomenon, using different terms, leaving one searching for a third language which would make everyone happy. What one does about a great deal of the data on the concept of space, contradictory to or requiring comment by Piaget's theory, is also largely a matter of choice. Given our current state of knowledge, the theory, if one believes in it, can be made to account for most all of the findings discussed here. Relevant systematic inquiry into Piaget's formulation has a relatively short history. And, the masterful theoretical structure created by Piaget will n ot stand forever, certainly. But the multitude of studies generated to clarify and investigate that theory is one of the marks of a "good" theory.

The direction of future inquiry is set by the critical concepts and methodological imperatives of Piaget's theory. Psychological research by and large has been, until about 1960, limited to psychological analysis of "states" rather than "change" and based on response choice rather than actions and transformations. For example, most studies of development of concepts of space are concerned with what cues are perceptible, i.e., is the child able to discriminate between a form or object and its mirror image. The important question, from the Piagetian perspective, is whether the child can identify an object that is a 180° change from the standard object. Or, more generally, can the child distinguish a change in position from a change in state? Thus, the strategies used by the child to distinguish various transformations constitutes the bases for describing developmental change. The main point of Piagetian theory is the shift from analysis of "features" to the coordination of input and action (Inhelder et al., 1966).

Each advance in science is accompanied by methodological and/or technical innovations made "realizable" by the new ideas. Piaget's contribution in this area in his refined observational procedures designed, as much as possible, to give the child opportunity for "spontaneous construction." Psychology has been a psychology of controlled choice and not a psychology of transformations. Piaget's epistemological imperative to psychology is that learning occurs by making transformations on objects and, further, that learning results from self-generated transformations:

> "In manipulating, besides learning something about the object in the course of such an experiment, the child also learns something of the way actions are coordinated and how one determines the other" (Piaget, 1968, p. 453).

The usual method of psychological research is to present successive exposure to stimuli controlled by the experimenter. The structuralist (Piagetian) approach, on the other hand, requires situations that permit self-regulation of patterns of action sequences. Of course, many psychological researchers believe that such procedures, while theoretically relevant, are too imprecise and not appropriate for the science of psychology. If, for example, input variables are determined by the "whim" of the subject how can "causes" be identified? The answer depends on our definition of "explanation." If the concept of "cause" involves the issues of "antecedents-consequent" relations, then the structuralist approach appears naive and out-dated. However, if formal (as contrasted to efficient) causality is accepted as a type of "explanation," there is no problem.

Research involving the psychology of transformations, including those concerned with space-time, require much finer grain analysis than typical of past behavioral studies. The microanalysis analyic of development à la Piaget is essential to further our knowledge generally and for testing Piaget's theory. Aside from the latter, microanalysis of action patterns under controlled conditions will fill in apparent discontinuities in cognitive development. That is, to discover and explicitly describe the nature of the functional interactions that gives rise to the hypothesized emergent functions.

As Forman (1973) suggests, microanalysis has the further advantage of placing the focus on the child rather than on the "task." It is not sufficient to know the developmental order of task; rather the experimenter must observe, record and analyze the *process* (action pattern) of task solution. Otherwise, the investigator can only make indirect interpretations; i.e., those derived from his *reasoning* of *what is logically required* to solve the particular problem. As example from Forman (1973): "it has been reasoned that the child can distinguish a two dimensional 'concavity' from a two dimensional 'convexity' by using the convention that shadows come from above (Yonas, 1973). Another reader could enter with his 'yea, but' and point to other features of the task which might have been the cues of response. If the researchers would take a closer look at the child, say with high-speed photography, they may find something in the nature of the research that would make the conclusions firm. What if the child extended a single finger toward the 'concavity,' as if to place his finger 'inside,' but when approaching the 'convexity' he maintained an infantile thumb-forefinger opposition as if to pinch at the 'protrusion.'" That microanalysis of behavior can add significantly to the understanding of spatial development is closer from the work of Forman (1973) and the study of visual scanning patterns by Vurpillot (1968), among others.

Emphasis on transformation capability, microanalysis of behavior (i.e., logic of action patterns) and open-ended response conditions all require a shift in our basic paradigm about experimentations (Smock, 1973). The full meaning and implications of this Piagetian "revolution" is only now becoming clear (Smock and von Glasersfeld, 1974).

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MAP Publications--September 1975 complete Update

Research Reports

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Research Reports

16) Radical Constructivism & Piaget's Concept of Knowledge
E. von Glasersfeld, June 1975

17) The Concepts of Appearance, Disappearance, and Reappearance: A Developmental Semantic Analysis
Michael Tomasello and Ernst von Glasersfeld, August 1975

18) On a Model for Learning and Teaching Mathematics
Leslie P. Steffe and Charles D. Smock, June 1976

19) A Constructivist Model for Instruction
Charles D. Smock, June 1976

20) Piaget's Thinking About the Development of Space Concepts and Geometry, Charles D. Smock, May 1976

Needed Research on Space in the Context of the Geneva Group

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Charles D. Smock, Director

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Preface

The Mathemagenic Activities Program is a model for early childhood education, sponsored by the National Follow Through Program, based primarily on Piaget's theory of cognitive development. The emphasis on structure of cognitive operational systems (logico-mathematical experiences) as the basis for understanding intellectual development provides mathematics educators with a unique opportunity for potential contributions to child development. This paper explores some of the implications of Piaget's theory for mathematics learning and teaching.

C. D. Smock, Director

Needed Research on Space in the Context of the Geneva Group

Jacques Montangero

and

Charles D. Smock

Research on space concepts needed to complement that which has been completed in Geneva should involve those aspects that have been overlooked as a result of the inherent limitation of the Piagetian approach. Piaget's approach has permitted an immense step forward in the science of psychology. We do not want to imply that Piaget should have considered all the points and problems which remain to be studied. His method, was, and is, the best suited to the study of basic epistemological problems as far as space is concerned. We do not question the value of the prior work by Piaget and collaborators; rather, if something new can be added to this work, it will be possible only if there is a change of method. The method suggested is not entirely different than the Piagetian method, but is an adaptation of the problems that remain to be studied. It needs to be stressed also that the structural approach of Piaget is inseparable from a developmental approach that is to be constructivistic and to imply equilibration processes.

The Piagetian explanation of spatial notions in children is mainly based on the concept of adaptive equilibrating structures. A structural explanation has the four main advantages that will be listed below before discussing certain implicit limitations.

1) The genetic structural approach does not deal with the superficial level of behavior (i.e., the level of "performance"), but sheds light on the underlying mental activity. For example, instead of giving a mere description, such as "some children can construct a tower the same height of a model tower because they know how to measure heights," the structural analysis endeavors to give account of procedures for knowing, i.e., the operations of partition, seriation, iteration, etc.

2) This type of explanation thus gives an account of the operational nature of knowledge. A concept is not described as a more or less static idea, but in terms of mental operations which consist in transforming what the subject observes (e.g., imagining the reverse of the action observed, or finding different ways which would lead to the same result). These transformational capacities enable the child to apprehend the transformations of the outer world. It is only when children can begin to understand physical or spatial transformations - by knowing what changes and what remains invariant - that one can be sure that they have reached an operational level.

3) The interrelation between the different inferences of a child dealing with a certain type of problem appear clearly. For instance, the different arguments a child gives in order to justify a conservation judgment correspond to different operations all within the same cognitive structure. The explanation of spatial abilities in terms of systems of transformations sheds light for example, on the close relation between different achievements of 7 year olds since these achievements all derive from the capacity to establish topological relationships.

4) The structural method points out the ressemblance between different fields of knowledge that seem heterogeneous when not studied with this perspective. The similarity of spatial abilities, such as length measurement, and logico-mathematical abilities, such as conservation of small numbers, appears through an analysis of these behaviors in terms of the grouping of operations. On the other hand, a precise description of the differences between notions can be given, e.g., of what is not alike in space measurement and number, or between topological and projective spatial notions.

The genetic structural approach also implies a search for what is common to children of a same developmental level and what is common to different ideas or judgments. Because of this particularity of intent and the generality of the structures used to give account of children's cognitive processes, the approach leaves some gaps in our knowledge of the psychology of space. The main limitations of the Piagetian approach to the study of space appear to be the following:

1) The role of the object of knowledge or what the subject assimilates as external reality does not appear clearly from previous studies. Though Piaget's theory stresses the fact that knowledge stems from the interaction between the subject and the object, the role of the object seems most unclarified (See: Epistemology and Education: A Collection of Essays, Smock, C.D., and von Glasersfeld, E., 1974; Radical Constructivism and Piaget's Concept of Knowledge, E. von Glasersfled, August 1975). A careful analysis should be made of this interaction; this means that the parameters of a situation must not

be studied per se, but with reference to the conceptual framework of the subject facing the situation.

2) Piaget's approach gives account of a very limited number of stages of cognitive development. It is necessary to describe many more substages in that development of order to clearly understand the process and to gain better knowledge of the different notions constructed during childhood and adolescence.

3) The mental operations forming grouping or group structures give a model of the abilities of children, i.e., all the possibilities of reasoning they have, but not the actual process (strategies) of solving problems. A study of the process itself would clarify the interrelation between the child's cognitive structures and the attributes of a situation; also, it would give invaluable information for educational purposes.

4) A metaphor can best specify the results of the cognitive studies conducted in Geneva; a forest of scattered trees, each tree representing a particular concept has been found; we now need beside studying the growth of more concepts, an understanding of the connections between these concepts. These types of studies are particularly important since there is strong evidence that the relations between different notions (or fields of knowledge) are major factors in promoting intellectual development (Piaget, 1974; Inhelder, Sinclair, and Bovet, 1974).

To make the prior body of research on space more complete the next phase should use some of the new methods developed in Geneva and should be applied to the study of this domain. In addition, types of experiments that have not been conducted in Geneva are necessary. The aim and the methodological principles of three main types of research will be suggested.

I. Interaction of cognitive structure and context variations.

The aim of this type of research would be to determine how the child assimilates different aspects of space; i.e., which spatial parameters influence his spatial judgments. The procedures would include both techniques for facilitating correct answers and, on the contrary, eliciting errors by providing material which may stimulate the development of spatial notions. One example relevant to these problems (i.e., varying the parameters of the situation) are the experiments on the concept of duration by Montangero (In press). A summary of this experiment is presented in order to exemplify the method suggested.

In his book on time (1966, 1969) Piaget demonstrated that operational judgments of duration are based on the speed of observed changes and the amount of "work" done (distance covered, number of objects handled, etc.) during the duration which is evaluated. In the more recent study it was hypothesized that, in addition to considerations of speed and work, the relationship established between the relative temporal order of starts and stops[1] played an important role in operational duration judgments and in their elaboration. One of the experiments designed to test this hypotheses, as well as to define stages of the development of the concept of duration, consisted in varying the parameters speed, work done and temporal order. The design is summarized in table 1.

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[1] For example the awareness that two events began simultaneously but one of them ended before the other.

Table 1

Summary of Design for Studying Variation in Context and Judgments of Durations[1]

	SPEED and WORK DONE				
RELATIVE TEMPORAL ORDER	No cinematic aspect; turning lamps on and off	Speed consists in frequencies; discrete actions; putting beads into a container		Speed consists in displacements: continuous movements of dolls or toy cars	
		Perceptible results: glass container	Non perceptible result opaque container	Distance covered comparable cardboard tracks	No permanent trade of distance covered
Synchronism: events start and stop simultanesouly	1	4	7	10	13
One event starts before the other; simultaneous stops	2	5	8	11	14
Same duration, no simultaneity: same time interval between relative starts and stops	3	6	9	12	15

[1] Number refers to the different experiment conditions.

Each subject was presented with the different situations obtained (15 situations in the above table) and was asked to give a relative duration judgment[2] and to justify it. The variations of the duration judgments and of their justifications were then related to the parameters of the situations in order to see what kind of relation the subject introduces (or constructs) between the different parameters. As already stated, the goal of such an analysis is not to study the importance of the parameters *per se*. Indeed, the results of this experiment indicated that a same situation is often evaluated very differently according to the stage of cognitive development of the subject. The presence of a certain parameter (e.g., a difference of speed without difference of work done) may have no influence on younger children's judgments but may become an important cue for the older subjects when they evaluate duration. Therefore, the fact of comparing the subjects' answers for different situations does not exclude developmental analysis of comparison of answers at different levels of development.

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[2] Answer to the question: Did the two events take the same time or did one of them take a longer time?

The analysis in term of relations established by the subject permitted a better understanding of the child's reasoning about duration; different modes were used to solve the temporal problems and the possibilities and limitations of the children's judgments at different substages of conceptual development were assembled at the same time. Similar methods applied to spatial judgments should yield the same kind of information about space concepts. The varying parameters could be different geometrical relations or different types of presentation allowing for different ways of apprehending similar relations. Such a method would be particularly suited to the study of the interrelation between "physical space" (spatial properties of objects discovered through active observation) and "logico-mathematical space" (spatial properties constructed by the subject and discovered through deduction).

The method of varying the context sheds light on what underlies each of the child's particular judgments. Comparisons can be made between contexts with other judgments _and_ with behaviors observed at different levels of development. These comparisons facilitate discovery of the status and causes of apparent regressions in the intellectual growth. For example, in the study of time, three types of regression were observed, which in every case, obscured actual progress in cognitive capability:

1) The early correct answers do not indicate the presence of a real understanding of the problem; they stem from a simplifying assimilation of the problem. For example, 5 year-olds tend to judge duration from one aspect of the situations only, namely the final temporal order. This simplification sometimes lead's to correct answers which, actually, are not duration judgments and therefore should not be considered as correct answers.
2) At a certain level of development, children may encounter more difficulties in solving a problem than they had previously because of the progress in an element (concept) of the "field" of ideas related to the specific concept study. For example, as the concept of length progresses, a temporal situation becomes more complex for the subject; i.e., he now considers the whole distance covered by the moving object, whereas he only took into consideration the places of starts and stops before.
3) Apparent regression may also be due to the fact that children vacillitate between two different modes of evaluation to solve a problem. In the duration studies, children often would alternate between a mode of evaluation based on the comparison of temporal orders and one based on the spatio-cinematic content of events. These altering centrations explain the fact that certain situations,

correctly evaluated by a child, using (for example) comparison of temporal orders, are incorrectly evaluated a few months later because of exclusive centration on other parameters. All such regressions disappear as the evaluations achieve the operational level.

The question of regression is discussed here because in several Piagetian spatial tasks a depression of performance is observed around the age of 9. Yet cognitive progress appears to be continuing. Thus, these regressions should be investigated and the method of context variation appear the best method for teasing out the relevant variables.

II. Interaction between spatial and logico-mathematical operations.

Geometrical abilities can undoubtedly help to understand mathematical problems. On the level of scientific thought, it seems that mathematicians have often recourse to spatial intuition (Beth, 1961). As far as child thought is concerned, the research completed in Geneva shows that spatial and mathematical notions are undifferentiated in young children (which is one reason for the failure to conserve small numbers). Later in development, the two types of notions are dissociated, but their structural aspects are isomorphic (Leskow and Smock, 1970; Alonzo, 1970; Bruce, 1968) without a rigorous deductive method, with the aid of imaged representation (i.e., spatial intuition) in the child when mathematical intuition is very limited. Consequently, it is not unexpected that many teaching methods on mathematics rely heavily on a spatial representation of mathematical problems.

On the other hand, geometrical abilities, though they are based on specific spatial operations, also depend on general logico-mathematical operations such as seriation, or representation of all the possible combinations, etc.

The possibility of using a systematic method is closely related to the logico-mathematical achievements of the formal operational stage (Inhelder and Piaget, 1958). When children reach this developmental level, they are able to apply such systematic methods to spatial problems (Greco, 1964; Bang, 1965; Leskow and Smock, 1970; Alonzo, 1970).

The study of the relations between logico-mathematical and spatial concepts should not overlook either of the following points:

1) Influence of progress in geometry on logical or mathematical notions.
2) Influence of progress of logico-mathematical notions on the development of spatial notions.

The bi-directional influence could be studied by methods comparable to the learning studies concerned with the reciprocal influence of mass conservation and class inclusion reported by Inhelder, Sinclair and Bovet (1974). Progress in inclusion (comparison of the sub-class and the total class) obtained through training induced improvements in conservation of mass, but the reverse did not hold. It would be very interesting to test, by a similar method, the reciprocal influence of mathematical and spatial acquisitions which normally take place approximately at the same time in cognitive development change.

The first step of such research would be to find suitable training procedures, i.e., exercises which are efficient in accelerating the acquisition of a concept because they take account of the laws of development.[1] The elaboration of a suitable training procedure for a concept goes through the following steps:

1) Cross sectional developmental study to define the different components (i.e., the set of inferences and operations) and stages of acquisition of the concept.

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[1] Development is here considered according to Piaget as a gradual coordination of schematas achieved due to the child's organizing activity and, always, an equilibration process.

174

2) Identify the different stategies that children use to try to solve a problem involving the concept. For conservation tasks, for example, Inhelder, Sinclair and Bovet presented the problem sometimes as discontinuous, and sometimes as continuous. The authors could distinguish in the young child two ways of evaluating the relative length of segments:

   a) by comparing the extremities of the segments (ordinal evaluation based on the "frontier" effect).

   b) by counting the number of discontinuous elements forming the segments.

3) Devise exercises comprising actions, or preoperational inferences, or operations that the subjects can perform. This implies that the optimal level of development of the subjects has first been determined and the operative capabilities of the subjects are known.

As the main goal of a training procedure is to stimulate the subjects mental activity vis-a-vis a particular problem, the best procedure consists of presenting different situations or types of exercises under conditions that permit comparison among those situations. In the learning study of length mentioned above, subjects had to perform the same task (constructing with matches a segment whose length had to be equal to the length of a model segment) in three different situations. The first tended to solicit an ordinal comparison of length (evaluation type (a) above) while the second situation tended to promote a numerical comparison (evaluation type b) and the third allowed for the discovery of the difference of unities used in the model segment and in the child's construction. The experimenter asked the subject to return to the first situation immediately after he finished the second task, and to return to the second and first situation after the third task was completed. (the three situations always remained in the child's perceptual field.)

Such a procedure, which permits comparisons between different ways of evaluating a parameter, often elicits conflicts between these modes and consequently leads to important improvements in the comprehension of the problem.[1]

Space and time, which are reunited in a single concept in relativist physics, are interrelated during the course of intellectual development. Piaget's books on time (1966, 1969) demonstrated that the operational notion of duration involves a consideration of the distance covered related to the speed. On the other hand, the notion of length seems to be first grasped due to a representation of displacements; thus, Piaget, defines space as the coordination of displacements. At least two kinds of problems concerning the relation of space and time should be further investigated.

The first concerns the notion of distance and its interaction with the concepts of speed and time. The research conducted up to now in Geneva showed how these concepts become progressively coordinated near the beginning of the concrete operational stage (between the age of 6 and 9). The acquisition, around 8 to 9 years, of an operational notion of duration allows for correct temporal comparison between different events, but 14 year-olds still encounter great difficulty in establishing correct relations between distance, time and speed, when the two latter parameters must be deduced from a spatial datum in a double reference system (Crépault, 1975).[2] No research evidence is yet available, however, concerning the effect of progress in cinematic notions on the concept of distance, or other spatial notions. This effect, therefore, should be investigated thoroughly.

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[1] To these main principles for elaborating a training procedure must be added other considerations, such as the necessity to evaluate as precisely as possible the subject's developmental level (pre-test) and the status of the progress observed (generalization at the post-test; stability of acquisition or consolidation at the second post-test), etc.

[2] A vibrating stick prints dots on a strip of paper moved by the experimenter. In one situation, the frequency of the stick is even, whereas the strip of paper is moved first slowly, then rapidly. The subjects are presented with the printed paper and must evaluate the relative time and speed of the displacement of the strip of paper between two dots close together compared to two dots at a wide internal (end of the strip).

A second type of experiment is needed to determine how children can understand spatial relations (relative distance, directions) in situations involving points of references, or observers, moving at different speeds. Here again, spatial relations would depend on cinematic cues. This type of problem - which could be understood only at the formal operational stage - is now under investigation at Paris VII University by a team of physicists (Saltiel et al) interested in the psychological foundations of physical concepts. One situation presented, for instance, consisted in imagining two parachutists falling from a plane at different speeds; one of them drops his eyeglasses that the second one manages to catch. Questions are asked about the duration, distance, speed of the fall of the eyeglasses relative to both parachutist number one and to number two.

Cinematic situations are interesting for the study of spatial relations even when no questions are asked about the temporal and cinematic parameters. Many experiments could be designed on the theme of trajectories of moving objects in a plane or within a three-dimensional space. A first example of such experiment is Inhelder and Piaget's (1958) research on the quality of the angle of incidence and the angle of refraction in a game of "billiards". In this experiment, there is a close connection between physical relationships (causality relationship of the position of the propeller with the impact of the ball thrown on the aim, which consists in a wooden block) and spatial relationships (the subject can understand the physical law only by spatially structuring the game board and by comparing the angle of incidence and the angle of refraction of the ball trajectory). Similar experiments, requiring comparison of geometric figures, could be done within a football game context. The study of trajectory should provide many suggestions of experiments where both the physical and logio-mathematical aspects of space are involved. In a recent study by M. and I. Fluckiger (Research on reflective abstractions. Piaget, J., in preparation) subjects were asked to imagine the trajectories of a tennis ball hanging by a long thread fixed to a hook in the ceiling. First, the children were required to anticipate the trajectory of the ball when the experimenter, who had removed the ball from the vertical axis where it normally hangs, would release it from a predetermined place. The question was: Where should one place a wooden pin so that the ball would knock it down? At the end of the experiment, children were asked to imagine the bowl-like figure (i.e., a portion of a sphere) formed by all the possible displacements of the hanging tennis ball.

III. Research relevant to applications to education.

Some people, incorrectly assume that results of the research in developmental psychology conducted in Geneva allow for direct applications to education. Those who understand the Piagetian studies know very well no such direct application to education is possible. An "Intermediate" body of research is necessary for purposes of application. Piaget has endeavoured to find out the general processes of knowledge acquisition or how better forms of knowledge evolve from early more limited forms and what underlies our basic scientific concepts. Thus, he has had to make abstraction of the individual characteristics of his subjects. Due to these abstractions, general developmental processes (such as equilibration) were discovered and the potentialities of children at different levels of development were defined in terms of cognitive structures.

When it comes to the elaboration to "good" teaching methods, the knowledge of these processes and potentialities is of great help, but certainly does not suffice. Teaching deals with unique individuals and therefore, it is necessary to base teaching methods on the results of research aimed at specifying individual cognitive characteristics and at describing the actual process (performance strategies) involved in the utilization of cognitive structures. Two kinds of research are most helpful to serve these purposes.

First, differential studies should be conducted, not only to show that there are differences in the acquisition of spatial concepts, but also in order to analyze those differences. One question which must be dealt with is: Are there different ways of solving a given spatial problem in children of the same developmental level? As a matter of fact, even differential studies of the usual type can give useful information for establishing teaching curricula.

The second type of research needed in view of educational applications can be based on the analysis of the children's organization of action in order to solve spatial problems. A team of research workers directed by B. Inhelder is undertaking such a study in Geneva, without focusing on spatial problems. This body of research is an attempt to shed light on the processes of discovery in action and on the interplay between the processes of goal oriented activity and the child's concepts and representations. An experiment involving both physical and spatial notions by Montangero (Organization of actions in a task of raising a water level by immersing objects) is another example. The material consists of objects similar in shape (cubes or balls) but of different weight or different volume. There are, for instance 5-face empty cubes and normal "closed" cubes loaded with plasticine; or plasticine balls and leaden balls of same diameter. Two glass containers are half filled with water and the child is asked to put whatever he likes into the water in order to find out which objects noticeably raise the water level and which ones have little or no visible effect on the water level. In a second part of the experiment, the water level has to be raised up to a certain point by immersing objects. This point cannot be reached unless the children bring some changes to the material, namely by loading empty cubes with plasticine or with a leaden ball.

The actions observed and the few comments made by the subjects are analyzed with the aim of specifying:

a) How sequences of actions can be delimited by considering the successive immediate goals of the subject.

b) What are the relations between the different sequences.

c) What can explain the changes of goals and of action organization.

On the one hand, the subject's actions are related, for each sequence, to the notions or particular representation which seem to direct the actions. For example, children up to 9 years of age, hypothesize that the weight is the cause of the raising of the water level, or constitute couples of objects, differing by one parameter, in order to compare their effect when immersed. On the other hand, the analysis tries to specify which aspects of the situation (characteristics of the objects, effect produced, experimenter's intervention) are taken into consideration by the subject and how each influences the course of his activity. In this respect, an investigation asks a question which was often overlooked in the research on cognitive structures; i.e., what is the role of the object of knowledge. The general implications of the results of the water level experiment, briefly summarized, are as follows. First, there is a rather large variety of action patterns among children whose conceptual levels seem similar. Second, this type of experiment clearly reveals how the successive sequences of actions are guided by an interaction between conceptual representations and the different cues of the situation. Third, the discovery of a means to reach the goal, in the second part of the experiment, is often preceded by successive practice of the isolated schematas which, once they are combined, allow for success. Finally, the influence of the results of the child's actions (raising of the water level) is limited and depends on whether the anticipations are confirmed or not, and corresponds very much to what appeared in a previous study on goal oriented activity (Karmiloff and Inhelder, in press).

In the experiment discussed here, the subjects' conceptual level concerning the main notions involved (dissociation between weight and volume) was preoperational. In the field of space, similar research on how children use an operational spatial concept when trying to solve problems which require the acquisition of higher concepts would be most helpful.

These suggestions for research are, of course, very general. It is hoped that many research workers, knowing well the previous developmental studies of space, will transform these suggestions into actual experiments. The results of such experiments should be very beneficial to the teaching of geometry as well as for cognitive psychology generally.

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MAP Publications--September 1975 complete Update

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FIRST PRINTING

Teacher Strategy Manual
for Mathematics

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A guide for using the mathematics curriculum of the Mathemagenic Activities Program of the Georgia Follow Through Model.

September 1974

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Dear Reader:

Before plunging into this manual you should take a few minutes to look at the various parts of the manual. Notice that Part I, Overview, gives you a broad look at the Mathemagenic Activities Program. More specifically, the section headed Considering the MAP Program contains the answers to questions which are frequently asked about starting in the MAP Program.

As you read, you may become curious about the "Activities" of the Mathemagenic Activities Program. You can find many activities described in full in Part V, Objectives and Activities.

Notice that Parts II and VI are nearly the same in title and intent. Part II is designed to present only the main ideas, while Part VI explores the subject in more detail.

This manual is designed to be used either as a text in preschool workshops or to be read independently. Either way, careful study will pay dividends in understanding the philosophy and methods of teaching used in the MAP Program.

Good luck,

The Authors

TABLE OF CONTENTS

I Overview ....Page 1
Considering the MAP Program
William D. McKillip

II Rationale and MAP Principles - Basic Version ....Page 10
William D. McKillip

III Activity Teaching - Active Learning ....Page 25
The MAP Model for Teaching Mathematics
Designing Activities for Children
Planning Activities for Children
Conducting Activities with Children
Edward J. Davis

IV Active Learning in the Classroom ....Page 39
Suggestions for the Teacher
Michael L. Mahaffey

V Objectives and Activities - a sampling ....Page 51
Kindergarten
Grade One
Grade Two
Grade Three
Edward J. Davis
Michael L. Mahaffey
William D. McKillip
Leslie P. Steffe

VI A Learning - Instructional Model ....Page 134
for Mathematical Concepts
Rationale and MAP Principles - In
more Detail and at a Higher Level
Leslie P. Steffe

VII On the Child's Conception of Number ....Page 161
Leslie P. Steffe

Chapter I

Overview

Considering the MAP* Program

It is fair to assume, because you are reading this manual, that you have some interest in the Mathemagenic Activities Program, or possibly that your school system is considering adopting it. When a school system considers a new program, most teachers asked... "What, in a nutshell, is this program? ... Will this change improve my teaching? ... Will this be a 'fad' to be adopted today and discarded tomorrow? ... Will this approach work in my classroom? ... Will the children learn more using this program than they did before? ... Does this approach require more time than I can give?"

Before presenting the details of the program, the psychology behind it and the way it works, we will attempt to give encouraging, yet honest, answers to these questions.

What is the MAP mathematics program? In this program, small group and individual activities are *gradually* substituted for whole class instruction. Some large group instruction will, of course, continue. Some activities are teacher led, while others are independent of teacher direction. All activities are designed so that children can carry them out in different ways, thus allowing for differences in development and ability. The program covers the same content you are now teaching but makes increased use of collateral materials, both physical material and supplementary units. It is a program of *method* and a *philosophy* of teaching; it is not a new mathematics curriculum.

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*Read this "Mathemagenic Activities Program" or just say "map."

Will this be a 'fad' to be adopted today and discarded tomorrow? We don't know; it could be. Some systems have been using this approach for several years; many teachers have used this approach, or parts of it, even without formal adoption by their school system. The commitment of a school system to implement and maintain the program and the efforts of teachers in making it work may well determine whether it is a fad or a permanent improvement.

Does this approach require more time than I can give? No. The time and effort required can be adjusted by moving into the program slowly or quickly. Teachers report that it does take extra time and study to get started, but not so much time that they can't handle it.

Will this approach work in my classroom? If, after you have studied this manual and had the inservice training program, you *want* to try this approach, it will almost surely work for you. If you are being "forced into it" or if you are convinced it won't work, it won't! In short, teachers can make this approach work if they believe it is of benefit to children.

Will this change improve my teaching? Will the children learn more this way? The answers to these questions are almost certainly yes. Using this program will incorporate proven educational principles into your teaching procedures without forcing you to abandon the techniques you are already using successfully. Teachers who have used the program tells us that the children learn more basic mathematics in a manner suited to their age and stage of development.

The Emphases in MAP

Every new program is invented to solve some educational problem or improve some teaching practice. The Mathemagenic Activities Program, also, was developed for the purpose of improving teaching and learning in some specific ways. The MAP program stresses active learning, some individualization of instruction, and particular modes of instruction which are closely matched to the child's stage of psychological development. We may emphasize here again that the content of your mathematics program will change very little.

Learning Through Activities. Underlying the "activity" approach to learning, you will recognize a conviction on the part of the authors that children learn by doing things. This is not a new idea. "Learning by doing" as an educational approach has never really been out of style with primary teachers! Children, and most particularly young children, learn mathematics by becoming actively involved in constructing the mathematics for themselves. Indeed, the reason for using many physical materials in teaching is to stimulate children to construct their own mathematical ideas and facts.

The activity approach, the "learning by doing" strategy, is used to learn mathematics. In this program, the children learn sound mathematics, arithmetic concepts and skills and geometric concepts, through activities specifically designed to produce mathematical learning. The activity approach is not a "play school" or an undirected set of free-choice experiences. The activity approach is a program of active participation by children in specific educational experiences. The activity approach rests on the conviction that children learn best when they do something; "remembering what the teacher said" is an ineffective learning avenue!

Learning proceeds through what children do. Teaching by telling is not generally effective.

When an activity approach is used in teaching arithmetic, the children generally work with a wide variety of physical materials. This is true in the MAP mathematics program. The children will use in their activities both teacher made materials and commercial materials. The use of a wide variety of materials is a necessary and desirable part of an activity approach to mathematics at the primary level.

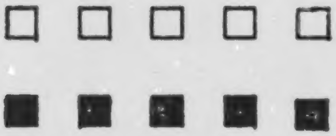
Individualization. Children differ from one another in many ways -- in home backgrounds, physical characteristics, growth rates, personalities, and in their potential for learning mathematics, to name only a few. How to adjust children's learning experiences when planning and implementing a school program has been a problem for teachers because a school program which is "just right" for one child will not be suitable for another. The search for a single curriculum which will somehow be "right" for all children and so "solve" the problem of individual differences is hopeless; no such curriculum can be found.

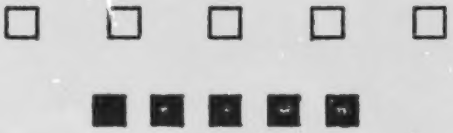
The MAP Program encourages the individualization of instruction. Recognizing that there is no one "right" curriculum, the MAP Program plans for children to learn at different rates and in different ways. This is done in two stages. Beginning with a total class, small groups are established. These groups are flexible and children may move from group to group as their progress indicates. More important than this, however, is the way in which activities are presented within groups. Each activity is designed so that children can work at their own stage of development. In the next section you will see ways in which a child's stage of development can be a determining factor in his learning.

Stage of Psychological Development*

Children are different and these differences are vitally important to the classroom teacher because they lead to differences in what children can do and in how they can do it. Children are also different from adults in the ways in which they handle basic mathematical concepts. In the Mathemagenic Activities Program, children approach mathematics in three stages: Preoperational, transitional and concrete operational. These stages are an important way in which children differ; they may be the most important characteristics to consider when planning mathematics instruction for young children. Let us look at these three stages.

Preoperational Stage. A child who is at the preoperational stage will reason about quantity in terms of the appearance of sets of objects. These objects must be physically present and subject to his manipulation for him to reason about numbers and number relations. This child will not "conserve" number, matching relations, or number relations; even when objects are physically present he will make errors when arrangements of objects are changed. For example, such a child will assert that the sets shown on the left contain the same number of elements but that the sets on the right do not. This implies that for the child who is at the preoperational level, numerals will have little meaning unless objects are present to provide an interpretation of the numerals which he can grasp.





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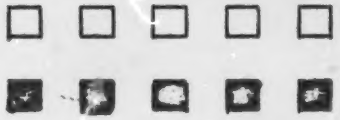
*A more complete description and analysis of this topic will be presented in later sections of this manual.

Many adults, even experienced primary teachers, are not familiar with this characteristic of young children. In fact, many teachers simply don't believe it when they read about it! You may want to try to demonstrate this characteristic for yourself, using a 5- or 6-year-old child as a subject. Try these simple problems. We use black and white blocks but any two kinds of objects will serve the same purpose.

"Are there more black, more white, or the same number of each kind?"



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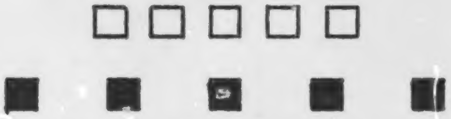
Ask the same question.



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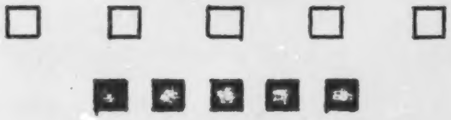
Ask the same question.



Do not conclude that a child at this stage is "slow," stupid or retarded. The child will progress to the transitional and, hence, to the concrete operational stage and this progress may be facilitated by school instruction. Some teachers attempt to substitute sheer memory for the understanding which should develop at this level. This practice produces a child who "knows his facts" but cannot use them because he lacks basic understandings about number. It also produces a child for whom arithmetic is a mystery to be memorized. If a child is in the preoperational stage, he should be allowed to work with objects for as long as he needs to rather than being forced to memorize.

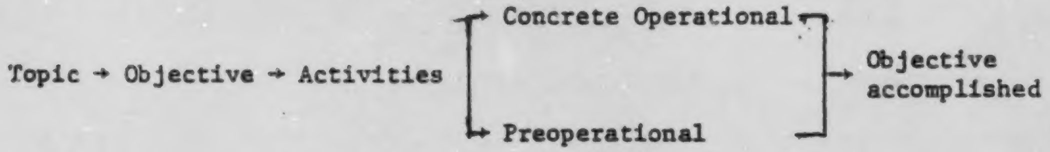
Concrete Operational Stage. A child at the concrete operational stage is capable of numerical reasoning which children at the preoperational stage cannot do. For example, children at the stage of concrete operations can reason correctly in the situation described above. In this situation a child will "conserve" number, matching relations, and number relations. In general, sets of objects need not be physically present for a child to reason about them correctly. Instead, the child can imagine objects and reason with his images. A child who is at the concrete operational stage will be able to work successfully with tasks which are presented abstractly; in fact, he may prefer to work in an abstract mode. He understands numerals and can reason correctly in situations in which numerals are used.

Transitional Stage. A child who is in transition from preoperational thinking to concrete operational thinking is said to be on the transitional level. This child will function at times on a preoperational level and at other times on a concrete operational level.

Optional Activities. We have noted that children differ in many important ways. One difference which is frequently overlooked is the stage -- preoperational, transitional, or concrete operational -- on which the child is thinking. We have noted that we are convinced that active learning, learning by doing rather than by listening, is superior to formal teaching. These ideas combine to produce the activity centered mathematics program of MAP.

Simply stated, the heart of the program is: If a child has a variety of activities from which to choose...if there are activities appropriate for children at different developmental stages...then the child will choose the activity which is appropriate for his stage of development.

This means that the curriculum structure is a series or chain of these units:



Read this diagram as follows: Given a topic we wish to teach, we state our objectives and prepare activities for concrete operational and preoperational thinking; each child may choose the one he "likes best," and follows the activity, accomplishing the objective.

Let us spell out a simple example which will be encountered at the first grade level. This topic is addition. We want the children to learn that three plus four equals seven. This defines our topic and the desired outcomes. Now we want a variety of activities, at the preoperational and concrete operational levels.

For the preoperational child, an activity would start with collections of objects which the child would form into sets of 3 and 4 and then combine these and count the number in the new set. After he has done this repeatedly with a variety of objects, they would discuss this action and then possibly symbolize it as $3 + 4 = 7$.

For the concrete operational child, the beginning of the activity could be the question concerning the open sentence $3 + 4 = \underline{\quad}$. The child would be free to draw pictures or use objects or to find the answer only through mental operations.

Each child is given a choice of how he wants to do the learning task, but the objective is basically the same for all. Thus, through his choices, each child will have a curriculum which is appropriate for his level of thinking...the child who is at the preoperational level can, as he advances, change to concrete operational activities.

Preview

As you read on, keep in mind the basic reasons for the existance of MAP. They are: To improve instruction by individualizing and by providing different avenues of approach to a topic for children at different stages of development; to encourage active learning and teaching styles.

In the following sections of this manual you will learn how to find and how to create activities which are appropriate for children at different stages of development and how to organize your class and your classrocms for the MAP curriculum. The next section is an introduction the psychology on which the MAP mathematics curriculum is based.

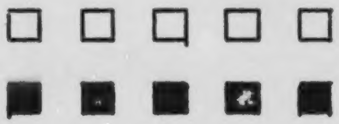
Chapter II

Rationale and MAP Principles

In this chapter, you will learn about two important factors in children's thinking: whether a child visualizes in "still" or "moving" pictures and whether or not a child can think about two things at the same time. You will see how a child's experiences contribute to his intellectual development and how the MAP principles are used to plan a child's experiences in the classroom. Finally, you will see a sample activity and learn how it is related to the child's developmental level and to the MAP principles.

Children's thinking at the preoperational level and at the concrete operational level: Children at the preoperational level do not perform "back and forth" mental operations; concrete operational children do.

We will now use an example which is probably already familiar to you. Two sets of blocks are arranged as shown in (1) on the right. The child, say a 5 year old, is asked, "Are there more white blocks, more black blocks, or the same number of each?" The child replies, "The same." Now the blocks are moved, as the child watches, to position (2). The same question is asked and the child answers, "There are more black blocks." In each case the child is responding to what he sees at the moment. When the row of black blocks is longer, it does "look like more." Position (1) is not considered when the child sees position (2).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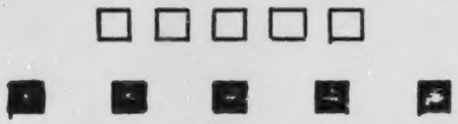

(2)

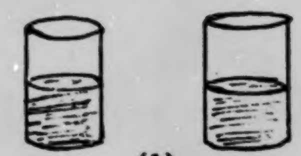
If we perform the same experiment with a seven-year-old child he will, very likely, respond that there are the same number of blocks in both positions (1) and (2). He will also think the experimenter has asked a silly question because he now *knows* "You didn't change the number, you just moved them around."

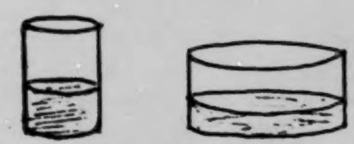
We can explain this difference in a way which will help us understand other aspects of children's thinking. Imagine that the mind of the five-year-old works like a still camera. The mind-camera takes a picture of the blocks in position (1) and responds "the same." The mind-camera takes a picture of position (2) and responds "more black" because in position (2) it looks like there are more blacks. The child now has two mind-pictures of two situations and responds differently to these two pictures. This does not bother the child at all because he does not compare the pictures. Because he cannot hold two things in his mind at the same time, he is never bothered by the contradiction in response which is so obvious to adults.

Now imagine that the mind of the seven-year-old child works like a motion picture camera, taking a "movie" which the child can run forward or backward. When the experimenter moves the blocks from position (1) to position (2), the child's mind-camera takes a picture of that act. The mind-camera then runs backward, reversing or cancelling out the move, and the child responds that there are still the same number of black blocks and white blocks.

To summarize what we have learned from this example, the mind of a preoperational child works like a still camera, taking pictures which the child examines (mentally) one at a time. He does not or cannot compare pictures to detect contradictions. He responds to each situation as he sees it. Because he does not *mentally* move the blocks around, he is said to be at the *preoperational* level. The mind-camera of a concrete operational child works like a movie camera, taking pictures which the child can run forward or backward. Thus, he can *mentally* perform an operation on concrete objects and *mentally* reverse or cancel it out (by running his mind-camera backwards). Because he performs these operations, he is said to be at the *concrete operational* level.

Look at a second example. In the first situation all children will assert that there is the same amount of water in each glass. They are the same because they look the same! When one glass of water is poured into a bowl (2) the preoperational child will say that there is more water in the glass. The glass has more because it looks like more! Observe these characteristics of his response: He has a "still picture" of each situation, he does not think about these pictures at the same time, and he responds to what the pictures look like.

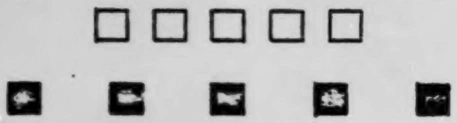

(1)


(2)

A child at the concrete operational level will take a "movie" of position (1), the act of pouring, and position (2). He will then run his movie in reverse, mentally changing from position (2) back to position (1) and continue to state that the quantity of water is the same in each container. Observe these characteristics: He has a reversible movie picture of the actions performed and so he can and does respond not to what he sees but to his picture of situation (1). "They were the same at first, and by pouring back they could be the same again, so they are the same."

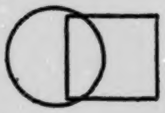
Children at the preoperational level do not consider two situations or two parts of the same situation at the same time; concrete operational children do.

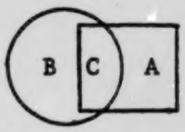
Think back over the first example about matching black and white blocks. One reason for the preoperational child's response, "More black blocks," to position (2) was his inability to think of position (1) and (2) at the same time and see the contradiction in his two responses. We will now look at some other examples of this same factor operating in preoperational children.



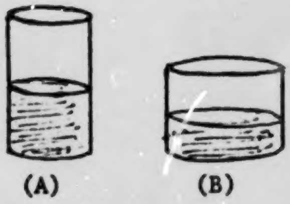
"More black blocks"
(2)

A child is given a block and ask to place it on the design on the right. The instructions given to the child are: Place the block so that it is inside the circle and inside the square." Notice that this direction has two conditions which must be considered at the same time. A child of five, at the preoperational level, will have a difficult time with this problem. Frequently a child will move the block back and forth from region A to region B as his mental focus shifts from one condition to the other. Seven year olds who are at the stage of concrete operations have no difficulty with this task. They can consider both conditions and locate the region (c) where the block should be placed.





Which container has more water in it? Most adults will hesitate..."Well, A is taller but B is wider, so I would say it



looks about the same." Preoperational children will select without hesitation either one or the other. If a child focuses on height, as most do, he will select A. "Taller means more." If he focuses on width he will select B..."Wider means more." During the years when a child is acquiring concrete operations, roughly from six or seven years on, he will attain the ability to think about both dimensions. Then he will see that "tall and thin" will be about the same as "short and wide."

We have seen that children at the preoperational level (1) cannot mentally reverse an action and return things to their original state and (2) cannot hold in their minds two thoughts at the same time. Children at the concrete operational level can do these things. We now go on to discuss the way in which the child's experience contributes to his development from preoperational to concrete operational thought.

The change from preoperational to concrete operational thought. The change from preoperational to concrete operational thought occurs as the child becomes able to mentally reverse actions he has performed and to hold two thoughts in his mind at the same time. This change takes place gradually, over a period of years. The child will attain the concrete operational level in various situations at different times. For example, the child will attain concrete operational thought for water pouring situations much later than for object-matching situations.

During the time of change a child will be inconsistent. At times he will appear to have achieved concrete operational thought and then he will revert to preoperational thinking. During this period a child is on the transitional level. We now look at the factors which influence the change from preoperational to concrete operational thought and what we can do about them.

Children learn through their experiences. In learning to mentally perform actions on objects, children must have many, many experiences in physically performing actions on the objects. It is, of course, not bad to watch others (teacher, other children) perform actions on objects, but we must be sure that each child has a wealth of his own experience. In addition to this personal experience in performing actions, the child needs to talk about his actions. When a child has performed an action on objects, he should discuss his conclusions with an adult. This is not a teaching situation in which the teacher points out errors. This is a conversation in which conclusions are discussed and further action, possibly leading to a different conclusion, may be suggested. Thus the child has two ways in which he is active: He is active in performing operations and he is active in talking about operations.

Another factor which has some influence on the child's progress is "maturation." "Maturation" refers to the physical and mental changes which occur purely as a result of growth. Within a broad span, maturation limits what we can expect of a child, but within this broad span experience is the potent determiner of the child's progress.

Let us imagine an example of a teacher and a child working with black and white blocks. The child and the teacher may both touch the blocks but the teacher would rather that the child perform the operations.

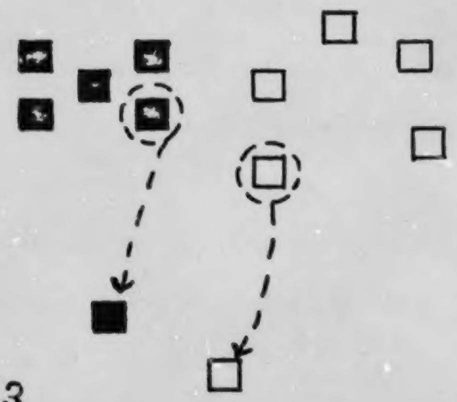
T: "John, are there more black ones or white ones or the same number of each?"

J: "More white."

T: "Can you pair them up, make pairs?"

J: No response

T: "Like this..."



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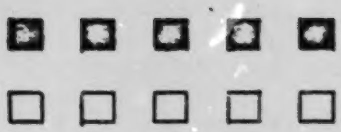
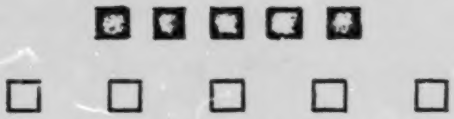
J: "Oh, yes..."

T: "Are there more black ones or more white ones, or the same number of each?"



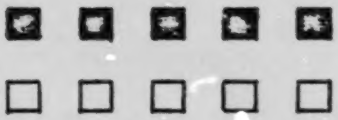
J: "The same."

T: "Watch this..." (Spread white blocks apart.) "Are there more black or more white or the same?"



J: "More white."

T: "Can you put the white ones back like they were?"



J: "Yes..."

(Notice that here we get the child to perform physically the reversing operation we want him to perform mentally.)

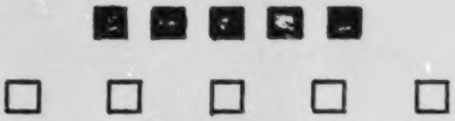
T: "Are there more white, more black, or the same?"

J: "They are the same."

T: "Yes, good."

Keep this example in mind as we draw some conclusions about the way in which such experiences can be effective. Insofar as possible, the child performs the actions and draws his own conclusions. We want the child to mentally perform these operations and so "teaching" here is getting him to perform them physically.

The teacher does not correct the child. When the child says "more white," the temptation is to say, "No, can't you see there are still just as many black? There are still the same

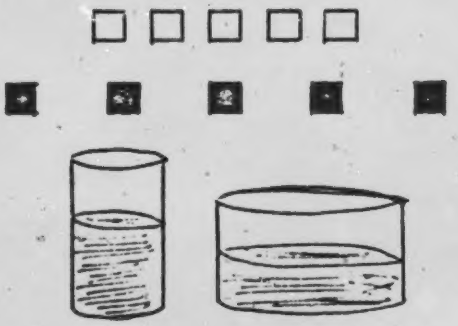


number of each." This does not help because, regardless of what the teacher says, to the child it looks like more white. Also, when the teacher makes the correction the child is not learning to perform reverse actions. However, the physical operation on the blocks *does* correct the child. When the child has performed the reverse operation, he again asserts that there are the same number of each.

In order for the child to achieve the concrete operational level of thought, he needs to have many such experiences. He does not learn from one or two such situations and teaching him to "say the right answer" is a fruitless procedure . But over time he has many such experiences which are seen and remembered. He takes these experiences into himself. As a result of this, gradually, he changes. He begins to perform *mentally* the operations he has done so often physically, and so these learning experiences contribute to the child's growth from preoperational thought to concrete operational thought.

The MAP Principles guide us in planning classroom activities. When you follow the MAP Principles, Activity, Mismatch, and Self-Regulation, you help children to develop at their own best rate.

Activity. To see the importance of the principle of "activity" we may think back over the last few pages. In these examples children are learning to perform mentally things which they can do with physical objects. We have seen that it does not help to tell the child that the numbers of blocks are still the same or that the quantities of water are the same. To the preoperational child, appearance is everything.



We can help him to perform mentally the reverse operation by suggesting that he perform this operation physically. He learns gradually through many repetitions.

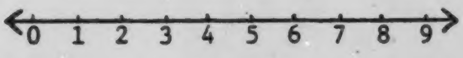
Now let us look at two examples of teaching. Neither of these examples is the "horrible example," but we hope you will agree that one is clearly better than the other.

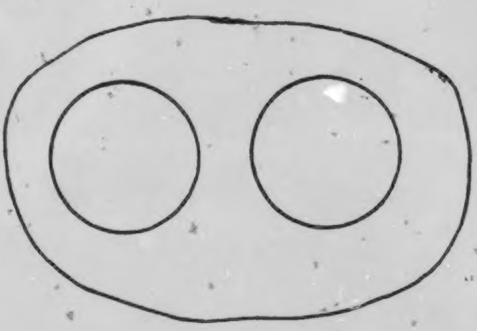
A teacher is teaching arithmetic, specifically the idea that two plus three equals five, to a first grade class. The children are seated, facing the teacher as she stands in front. The teacher asks first two children to come up, then three more. She points to the two and asks, "How many are here?" and then the same for the three. She asks the children to step together into a group and asks, "How many altogether?" The children count and respond "Five." The teacher says, "Right, very good, two plus three is five." The teacher then does the same thing with sets of books on her desk and sets of stars drawn on the board. She concludes the lesson by writing "2 + 3 = 5" on the board and reviewing the meaning of each symbol and how it is to be read. She may go on to present one or two more addition facts in the same way. Related practice work is assigned to the children to do. Not a bad lesson; in fact, we have seen lots worse!

Let us examine the assumptions that the teacher made in planning and presenting that lesson. The teacher assumed that all the children were ready for that lesson, that their previous mathematical background and level of operation were adequate. The teacher assumed that a small number of concrete examples would be sufficient for all children to grasp the meaning and significance of "2 + 3 = 5." The teacher assumed that the children could learn by watching and listening what they needed to learn from the lesson.

Now we will watch another teacher teach the same lesson. She is sitting at a table with eight children. Available at the table are bottle caps, Cuisennaire Rods, number lines, paper and pencils. The teacher begins with a question, "Pretend I have two bottle caps in this hand and three here. How many bottle caps do I have?" The response is mixed. Some children know the answer immediately, some don't seem to know it at all. Others begin to form sets of bottle caps. All children are invited to make sets of bottle caps and count to find the answer. The teacher then asks, "If I have a frog who goes two hops on the number line and then he goes three hops...how far has he gone?" Children may then use the number line to find the answer. Some knew the answer immediately, of course. The teacher works similarly with Cuisennaire Rods. She then goes back to counting objects and suggests different problems for different children. Some need to work on two plus three; others may be ready to explore three plus four or larger sums.

The teacher carefully teaches the meanings of the symbols the children will need to read. The numerals 2, 3, 5 and others, +, = and so forth, are reviewed. The teacher hands out a sheet of problems which some children work using objects to find the answers while others can find the answers without objects. A number line and/or a set diagram at the top of the sheet encourage children to use these aids when they need them.





This teacher made assumptions, also. She assumed that all the children could count and were familiar with addition as a combining of sets or as hops on the number line. She should have previously taught these things. The teacher did *not* assume that the children were at the same stage of learning: Some children were able to do the problem in their heads while others did a lot of work with concrete objects of various kinds. She did *not* assume that the children were able to learn by watching a listening. She provided for each individual to have an appropriate amount of concrete experience. Both teachers were careful to explain the meaning of all symbols and to teach the children how to read the mathematical sentences they would encounter on the practice work.

As you think about these examples remember...the first teacher taught through *watching* and *listening*...the second teacher taught through *active participation*. This illustrates the MAP Principle of *Activity*.

Mismatch. For a child, or an adult for that matter, to grow and improve there must be a difference between what he has presently achieved and what he is trying to achieve.

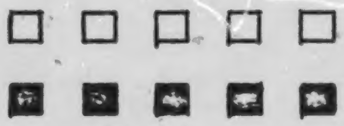
When a high school boy runs the mile in 4 minutes 30 seconds, the coach may say, "That was fine...4:30 is a good time...but if you run the third lap a little faster and kick about 20 yards sooner, I think you can get down to 4:20." Now, this boy may eventually run a mile in 3:55. Growth, strength and practice will all contribute to that. But to suggest 3:55 as a goal now is unrealistic because it is unattainable. So the coach set a reasonable goal...4:20...and gave the boy some suggestions about how he might attain it.

That is what we mean by "appropriate mismatch" between what the child can already do and what you are asking him to do: The goal is attainable, just a step from what he can do now.

There are three kinds of mismatch: content mismatch, level of abstractness mismatch, and cognitive stage mismatch. A content mismatch occurs when you present to the child a problem which is just a step from a problem he has already solved. When a child has solved 2 + 2 = 4, he does not solve it over and over...he goes on to 2 + 3 = 5 and 2 + 4 = 6 and so on. Your best guide to this kind of mismatch is to follow the sequence of topics in your text book. The content mismatch has been very carefully worked out by the authors.

Mismatch in level of abstractness occurs when you know how a child is solving problems and then present a problem to him on a higher level than he is presently using. Let us reconsider some examples we have already described, looking specifically for mismatch.

Think about the example involving black and white blocks. Even though the child does not conserve the matching, he says "more black" in situation (2), the conservation task is presented. This invites him to move to a concrete operational response and sets up a situation in which he can practice the reverse operation. Thus this situation is a "mismatch" between his present level of operation and the level to which we want him to move.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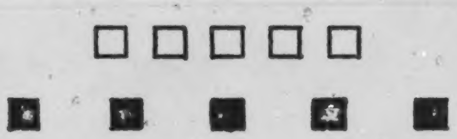

(2)

The teacher who taught 2 + 3 = 5 by sitting at a table with a group of eight children introduced the problem (the problem, of course, is what is the sum of two and three) by saying, "Pretend I have two bottle caps in this hand and three in this hand. How many do I have?" Presenting the problem with imagined physical objects invited the children to work out the problem through mental operations. The teacher knows that most of the children are using objects to solve such problems. She deliverately selected the "imagined physical object" mode for her question because it created an appropriate mismatch. We now look at this addition problem in an attempt to understand mismatch more clearly.

PROBLEM STATEMENT	METHOD OF SOLUTION	RESPONSE
"2 + 3 = ______" (verbal or written)	Knows answer instantly but can explain it if asked	"5" Instantly
"2 + 3 = ______" (verbal or written)	Counts; says or thinks "2, 3, 4, 5."	"5" Brief delay
"Pretend I have 2 bottle caps here and 3 here. How many do I have?"	Counts; uses sets of objects for counters or imagines sets of objects.	"5" Delayed
"2 bottle caps and 3 bottle caps make how many?"	Makes sets and counts.	"5 bottle caps"
"Make a set of two bottle caps and a set of 3 bottle caps. Push them together. How many do you have?"	Follows directions explicitly.	"5 bottle caps"

Children will start at the bottom of this chart, getting the basic meaning of addition by following directions. From then on the questions become more abstract, that is, less directly related to concrete objects. We obtain appropriate mismatch by asking the question on a higher level of abstraction, a level above the child's present operating level.

There are two other facts which must be considered. Sometimes appropriate mismatch is no mismatch at all. If a child has just recently moved from one level of abstraction to another, we should stay there until he is sure of himself on that level. Presenting a problem too far above the child's level will cause too great a mismtach and the child's progress will be slowed.

Appropriate mismatch is necessary for the child to progress. We should plan for mismatch between the tasks the child is able to do and the tasks we ask him to do. In summary, mismatch may be in subject content, level of abstractness of a problem, and in cognitive stage.

Self-Regulation. It is difficult for a teacher to be fully aware of the position and level of each of the children in her class. This is even more difficult if the teacher is encouraging children to progress at their own rate and to solve problems at their own level. Here the principle of self-regulation will come to the aid of the teacher. If activities which can be done at several levels are used, children will amost always select an appropriate level for their work. They will, in effect, regulate their own learning in natural and desirable ways. This is the principle of self-regulation.

Again, we refer back to the teacher who is using an activity approach to the teaching of 2 + 3 = 5. In her group there were children on many different levels. She presented the problem at a middle level of abstraction. Look back to the chart and note that the lowest level is specific directions for manipulating objects, the highest level is the arithmetic problem presented without reference to physical objects..."2 + 3 = _______."

The children were free to use whatever problem solving technique they wanted or needed to use. At this point in the activity the children regulated themselves by selecting an appropriate level of operation at which to work.

In Section V of this manual you will find a variety of activities. Most of these are multilevel activities with opportunities for self-regulation built in at appropriate points.

Summary. In this section we have considered two main topics. The first was the developmental stages of students, the *preoperational stage*, *the transitional stage*, and *the concrete operational stage*. The ways in which children can and do think is strongly influenced by his developmental level. The second topic was a discussion of MAP Principles: *Activity*, *Mismatch*, and *Self-Regulation*. Learning and understanding the MAP Principles helps us to plan appropriate instructional procedures.

Chapter III

Activity Teaching - Active Learning

What are the advantages of an activity approach to the learning of mathematics? The primary advantage is that it is a personal approach to learning. Mathematics, by its very nature, is an abstract, logical, subject. This is one of the reasons why some people dislike mathematics. Embedding mathematical ideas in activities in which students participate is a way of making the subject "come alive" or "make sense." Activities help make mathematics seem real for children and not so abstract.

Activities engage the child. They capture his attention and give him a sense of personal involvement. In an activity the child can ask specific questions about his own work and the teacher can answer these questions on a personal basis. To the extent that a student can self-regulate his choice of activities or to the extent that his activities are carefully selected, the child has the opportunity to engage in learning experiences in which he can be successful. Successful completion of an activity motivates the child to work on another task and contributes toward the positive growth of the child's self concept.

While we advocate the active approach to teaching mathematics, it is important to make the point that activities will not resolve all learning problems, be they conceptual, emotional, or social. There are no guarranteed solutions to educational problems! For example, for a child who is a nonconserver of number, exposing him to a dozen carefully planned activities, may not make him a conserver of number. Learning is personal. A teacher can "set the stage" by telling, drilling, engaging the child in activities, or in some other way; but it is the child who does the learning.

Before examining different types of activities, some of the statements concerning activities and active learning made in Chapters I and II are discussed. It was stated that: in the MAP program active learning increases while learning by hearing decreases. We also stated: children learn best when they do something. Both of these statements are important. However, it is also important to point out what these statements do not claim! Specifically, they do not state that learning by doing is the only way to learn; or that telling, or giving instructions, or discussions are to be avoided at all costs. Teacher talk and verbal interactions with pupils are necessary and desirable in the learning and teaching of mathematics. The points we wish to emphasize are: (1) learning by doing is very desirable, and (2) teacher talk is most effective when it is about something the child has done or is doing. In short, it is advocated that: whenever possible, provide children with active experiences and discuss those experiences.

Another statement made in Chapter I was - children learn by doing. This statement is correct as far as it goes but it needs a slight clarification. Children learn by thinking about what they are doing. It is not enough for a child to simply go through the physical motions of learning activity. To acquire mathematical knowledge, the child must think about what he is doing or has done. Often the teacher may have to help the child reflect upon his actions. This leads us into the heart of this section, namely, what is the nature of an activity that is appropriate for children whose thinking is preoperational? transitional? or concrete operational? Children in these stages think differently. Thus, if an activity is to have a student think and do, the activity must be designed, and the teacher's expectations must be influenced, by the child's thought processes and capabilities. We turn to these considerations in the following two sections: Designing Activities for Children, and Conducting Activities With Children.

Designing Activities for Children

Levels for Activities. In the following chapters of this manual, sample activity sets and a detailed discussion of a model (or plan) for teaching and learning mathematical concepts will be presented (Chapters V, VI, and VII). At this point an overview of these two sections is given. This is being done to provide the reader with the basis of the rationale for constructing different kinds of activities and a background for understanding the model for teaching mathematical concepts in Chapter VI.

Roughly stated the preoperational, transitional, and concrete operational stages of cognitive development are related to three levels of learning activities. These are exploratory activities, abstraction and representation activities, and formalization and interpretation activities. In Chapter VI a case will be made for restricting preoperational children to exploratory activities, engaging transitional children in exploratory and abstraction and representation activities, and have concrete operational children participate in all three levels of activities. A schematic representation of this recommendation is given below.

	Exploratory Activities		Abstraction and Representation Activities		Formalization and Interpretation Activities
Preoperational children should engage in	✓				
Transitional children should engage in	✓	and	✓		
Concrete Operational children should engage in	✓	and	✓	and	✓

Exploratory activities are for all children. They should begin the learning cycle for all mathematical knowledge. In an exploratory activity the child primarily performs actions on objects. His actions can range from free play (through which he can become acquainted with the objects) to following teacher directives for using materials in a specific way. In exploring with objects, a child will at least gain knowledge about the objects themselves — this will be called physical knowledge.

Abstraction and representation activities are intended for transitional and concrete operational children. These children can obtain more than physical knowledge from their actions. They can realize mathematical relations such as equality and begin to use conservation, transitivity, and reversibility to reason with the mathematical relations. Preoperational children also reason, but their reasoning is limited to objects and properties of objects and does not include reasoning with mathematical relations.

In an abstraction level activity a child is encouraged to realize a common element in various situations. The child is asked to see the essence of the actions he has performed on objects or representations of objects. For example, the teacher may have children packaging sets of objects into plastic bags, with the intent of having the children realize (abstract) the notion that any collection can be grouped into sets of ten with perhaps one set containing less than ten left over.

In a representation activity children are asked to use a symbol to stand for a particular idea or situation. Usually an idea he has previously abstracted. Representations include diagrams, sketches, tally charts, as well as mathematical symbols such as 15, < and +. Often, abstraction and representation will be part of the same activity.

In a formalization activity a child is guided in using representations (usually symbols) in a specific way. There are many rules that a child must use in working with symbols. In using the concept of place value, specific rules govern the values of each symbol in each numeral. The child must be aware of these rules and be able to use them. In the algorithms for addition, subtraction, multiplication, and division there are formal procedures that children are expected to follow. If a child has proceeded through exploratory, and abstraction and representation activities, he has a basis for understanding the rules or agreements presented in a formalization activity.

Perhaps some one of the frustrations that you encountered learning mathematics stems from your being placed in situations where you had to manipulate symbols rules before you understood the meaning of the symbols or the significance of the rules. It is possible to teach children to manipulate symbols in the absence of understanding. Many of us learned mathematics in that way. However, most teachers would prefer to have understandings precede or accompany rules. That is the basis of our sequence of activities. Exploration, and abstraction and representation activities provide the basis for understanding. These activities should precede instructions in using symbols in specific ways (formalization).

Interpretation activities call for children to work from symbols to objects. A child is given a symbolic situation, e.g., 3 X 5, and asked to construct a representation of what the symbols indicate. That is, the child interprets the symbols by constructing a diagram, arranging objects, by explaining his own words, or by using other symbols. The symbol "3 X 5" could be interpreted in various ways. Some of these include three groups of five bottle caps, the number of fingers on three hands, 5 + 5 + 5, or a diagram showing three sets of five objects. Interpretation activities serve as a checkpoint for the teacher. They provide the teacher with a way of evaluating the child's understanding of symbolic statements.

The cycle of activities for learning mathematics is now complete. Beginning with explorations and manipulations, the child then abstracts and represents ideas and relationships, and then is encouraged to follow certain procedures and to show his understanding of symbolic representation by giving mathematical or physical interpretations to symbolic statements. Schematically, children should go through the following cycle in learning mathematical knowledge:

First → Explore and Manipulate

Then → Abstract and Represent

And then → Use Formal Rules and Interpret Symbols.

Multiple Embodiment. Another consideration is offered for designing activities. Almost any activity, once constructed, can be modified. One way of modifying an activity, is to vary the objects used or the setting in which the activity takes place. We will call such a variation multiple embodiment. Applied to an activity requiring children to pair cowboys and horses to determine a matching relationship between the collection of cowboys and the collection of horses, the multiple embodiment principle would guide the teacher to find other sets of objects that could be paired. For example, crayons and pencils, or checkers and squares on a checkerboard. The multiple embodiment idea could also be employed in changing the setting of this activity. For example, pairing cowboys and horses, or checkers and squares is usually done in a small group setting. A teacher should be alert for matching in other contexts such as a situation where cookies and the entire class of children be placed in a one-to-one correspondence. The multiple embodiment principle is especially useful in creating adaptions of exploratory activities.

Mathematical Variability. A second way that an instructional activity can be modified is to vary the mathematical content of the activitv. We will call this form of modification mathematical variability. In a context of using the matching relations "as many as," "more than," and "lower than," mathematical variability can be employed to have children consider other relations on the same sets of objects. Relations such as longer or shorter, taller or shorter, heavier or lighter, wider or narrower, and older or younger can often be applied to the same objects. In a context where geomtric shapes are being used to develop the child's concept of congruence, a teacher might use mathematical variability to investigate other geometric ideas such as area or perimeter, or even use the geometric objects to review some previously studied mathematical ideas related to addition or multiplication.

In the course of presenting sets of activities for your consideration in Chapter V, we will occasionally mention how multiple embodiment or mathematical variability can be employed to modify a given activity. Multiple embodiment and mathematical variability are merely new terms for practices that teachers have been using for a long time. They are useful tools to employ.

Planning Activities For Children

With the above considerations in mind, let us examine a way one can construct sets of activities. Part of constructing activities involves insight and creativity. Exactly how one obtains insights and generates creative ideas is probably impossible to describe. However, there are some elements that were common to the construction of the sets of activities constructed and listed in Chapter V. First, an overall objective was determined. This is stated in very general terms. An example is: The child learns how to find the sum of any two single-digit whole numbers. Next, specific student behaviors or actions are listed. These specific objectives outline a series of behaviors starting with real objects and then proceeds to the introduction of symbols (representations), the manipulation of symbols (formalization), interpretation of symbols, and in this case, to memorized responses. These are listed below:

Sample Activity Set - Addition Grade 1

OBJECTIVE: The child learns how to find the sum of any two numbers. a and b where $a + b \le 18$.

A. Given two whole numbers a and b whose sum is ≤ 18, the child can select sets of a and b objects.
B. Given two collections of objects, the child can find the sum of the number of the collections by counting.
C. The child finds the sum of two numbers, a and b, by constructing sets of a and b objects and counting the total collection.
D. The child uses sets of objects to show $a + b = b + a$
E. The child uses objects or talley marks to solve equations of form $a + b = \square$
F. The child can construct (work) number sentences or the form $a + b = c$ to describe the union of two sets of objects.
G. The child begins to memorize the basic 100 addition facts.

In constructing specific objectives and in designing activities the teacher can ask herself GUIDING QUESTIONS. These questions follow from the levels of activities discussed earlier in this section. A sample set of such questions would be:

1) What objects, pictures or situations can I have the students use and explore to have them begin to meet the general objectives?

2) What are the basic ideas involved in the general objective? What can the students do to help them see (abstract) these ideas?

3) What symbols are used to represent the child's explorations and the abstractions I want him to make?

4) Are there any formal rules or agreements about these symbols? How can these be shown or illustrated?

5) What can the child do to convince me he *understands* (*interprets*) the rules or symbols he is using?

Questions similar to these above can help the creator of an activity set in formulating specific objectives. These specific objectives along with the answers to the questions form the basis for the construction of Exploratory, Abstraction and Representation, and Formalization and Interpretation activities.

Conducting Activities With Children

Placing a child in a situation that calls for a type of thinking that he can do is very important. Almost all primary classrooms have children at different stages of thinking, i.e., preoperational, transitional, or concrete operational. How do you *conduct* an activity in a way that is appropriate for a given child's level of thinking? The guidelines below are given to help answer this question.

Conducting Activities with Preoperational Children. An exploratory activity for a preoperational child should begin with, and continue to use, real objects (buttons, rods, blocks). In many exploratory level activities, symbols, e.g., 3, 10, +, =, will not be used. For example, in matching collection of objects to determine the appropriate matching relation (as many as, fewer than, more than) symbols do not have to be used unless you are using the activity to teach the meaning of numerals. If mathematical symbols are used, they should *come from and after* the manipulations of objects. Strictly speaking, if symbols are used abstraction and representation are present. Thus an activity which is basically exploratory can have some abstraction and representation aspects. As long as the abstractions and representations (symbols) play a secondary role no major conflict occurs.

In addition, the following should be built into the learning experience or into the way in which the activity is administered.

- Conversations between the participants and the leader about what they are doing.
- Manipulation of the objects by each child - preferably with each child having his own objects.

Let us re-emphasize a point made earlier concerning the nature of a teacher's conversation with a preoperational child. Preoperational children should not be expected to draw inferences and give correct answers to questions requiring conservation, transitivity or reversibility. This is not to say that situations involving conservation, transitivity, and reversibility should be avoided at all costs. But rather than children should not be expected to reason successfully in such situations. The teacher should make observations or ask questions such as: "Can you put them back the way they were?" or "Look, we can put them back just like they were by doing this." Such teacher initiated exchanges are desirable and can be considered as leading up to the notions of conservation and reversibility.

Conducting Activities with Transitional Children. An exploratory, or abstraction or representation activity for a transitional child may begin with real objects or pictures of objects. Symbols should be used and the child asked to relate the physical or pictorial situations to symbolic statements. In addition, if possible, the following should be built into the learning experiences or into the way in which the activity is administered.

- Questions that ask: Why? How do you know for sure?
- Questions that ask the child to predict what will happen if the situation is changed in ways that involve conservation, transitivity, reversibility.
- A graph or chart summarizing the activity.

Since a transitional child will be inconsistent in conserving quantities, in making transitive inferences, and in reversing his thinking, a teacher should encourage transitional children to think in these ways and have these children verbalize and check their mental operations. When transitional children make erroneous conclusions, a teacher *should ask questions* that are intended to help the child make a correction *and then discuss* the situation.

Conducting Activities with Concrete Operational Children. An exploratory, abstraction or representation, or formalization or interpretation activity for a concrete operational child may be oriented toward the use of mathematical symbols. Initially the child may be asked to relate the symbols to objects or to some known procedure. That is, at the outset of an activity a child should be able to explain what the symbols represents; but this relationship does not have to be explained over and over by the child. Activities for children at this stage of intellectual development can also:

- Require reasons and ask for conclusions or predictions that involve conservation, transitivity, and reversibility,
- Ask the child to think about more than one operation at a time, e.g., think adding 3 and subtracting 3 as doing and undoing,
- Involve drill,
- Ask the child to check his answers or work in concert with a partner in arriving at an answer, e.g., in measuring one child can hold the string tight while another applies the unit of measure -- each child can realize the significance of the others' role.

Three contrasts run through the above guidelines. The first is the role of mathematical *symbols*. For preoperational children, symbols play at most a secondary role. If symbols are used, they should be used after actions on objects have taken place. Symbols are used to label things -- after these things have been used or manipulated in some sense by the child. The most obvious set of symbols that will be introduced to preoperational children is the set of numerals (0, 1, 2, 3, 4, 5, 6, 7, 8, 9). At the preoperational stage of thinking, *objects dominate symbols*.

At the transitional stage symbols should be used and considered in concert with objects or pictures of objects. The teacher should be concerned with point out, and having children point out, the relationships between the objects and the symbols. For example, when transitional children combine or separate collections of objects, the symbols that represent these combinations or separations should be presented and discussed with the children. At the transitional stage *symbols and objects play equally important roles*.

At the stage of concrete operations *symbols play a dominant role*. The child can be asked to use objects to interpret what the symbols are saying. The child considers the symbols first, not after or along with manipulations of objects. Real objects no longer have to be present -- the child can imagine them or use other symbols (like tally marks) to stand for objects.

The second contrast running throughout the guidelines for conducting activities for children at different levels of thought concerns the concepts of conservation, transitivity, and reversibility. Preoperational children cannot think or reason along these lines and should not be expected to do so. They can, however, observe situations involving conservation, transitivity, and reversibility and participate in conservation, transitive, and reversing tasks led by the teacher. Over a period of time these experiences may foster the development of these concepts. However, the teacher must be aware that the preoperational child will not generalize what he observes in one activity to another situation. His learning is specific to the task at hand.

Transitional children are in the process of forming the concepts of conservation, transitivity, and reversibility and should be asked to make and then check predictions involving these concepts. These children need to observe and experience conservation, transitivity, and reversibility in a wide variety of contexts (this is referred to as multiple embodiment). The teacher should remember that these children are not yet "conservers" or "reversers" and they will often revert back to basing their conclusions on how the situation intuitively seems to them.

Concrete operational children have acquired conservation, transitivity, and reversibility and can consistently use these concepts in their thinking and to justify or explain what they have done.

The third contrast in comparing activities for preoperational, transitional and concrete operational children involves the nature of the conversation (questions) the teacher directs. Whenever possible, a teacher should refrain from correcting or pointing out logical errors that preoperational children make. Rather, the teacher should (if possible) have preoperational children perform additional manipulations and wait for the child to see the conflict. The waiting is essential! Even if it takes months! Preoperational thinking cannot be changed to concrete operational thinking by directions or rules. Preoperational thinking develops into concrete operational thinking as a result of the interaction of experience and maturation.

Conversations and questions with transitional and concrete operational should include an examination of the child's logic and conclusions. These children should be expected to justify their answers. If an error is made, transitional children can be helped to see the cause of the difficulty and to examine its cause.

We summarize these three contrasts in the following chart.

CONSIDERATIONS FOR CONDUCTING ACTIVITIES
WITH STUDENTS AT DIFFERENT STAGES OF THOUGHT

Children whose thinking is:

	Preoperational	Transitional	Concrete Operational
Symbols →	Symbols play a minor or secondary role. Symbols (if used) follow from manipulations of objects or pictures.	Symbols play an equal role with objects or pictures. Symbols should be an integral part of the activity. Encourage children to talk about them.	Symbols play a major role. Activity should start with symbols. Check to see if a child can relate symbols to objects-then the use of objects is optional (available if the child needs them).
Concepts of Conservation Transitivity Reversibility →	These concepts are not required for successful completion of activity. O.K. to present situations that incorporate conservation, transitivity or reversibility but do not expect or require a child to use them correctly.	These logical considerations should be used and discussed.	Child is expected to have incorporated these concepts into his thinking and be able to use them correctly.
Conversation →	Conversations should focus on objects. Use objects to help a child explain his answers. Use objects to try to correct any incorrect statements.	Encourage justification of answers. If errors are made try to have child realize his mistake. Teacher can use verbal explanations and point out errors the child has made.	

Chapter IV

Active Learning in the Classroom

Getting Started

There is no unique way of getting started in any activity approach. Most likely every teacher who has used "activities" in her classroom has begun in her own way and, while different, each may be good. While you may want to make your own plans for beginning, some general suggestions that teachers have found workable are offered here.

Initiating an activity approach to teaching is not an "all or nothing" proposition. A start may range from one small group to the total involvement of the whole class. Teachers should judge their own situation and select a starting point accordingly. It is suggested, in most cases, that a gradual beginning be utilized. Beginning with a small group of children (4 to 6) has the advantage of allowing you, the teacher, to perfect your methodology while at the same time providing for interaction by the children and discussion of the activities provided. While it is not necessary, it is logical for you to make your start with your better children.

When utilizing small group activities, you are confronted with two problems: (1) supervision of learning activities for the remainder of the class; and (2) expanding the small group activities to class activities. In the beginning, explain to the entire class what is happening and make it known that all will participate in some small groups. With close planning (with aides, if available), you can provide instruction for the remainder of your class and still supervise and guide your pilot group. In the beginning, you may want to use one day a week for your activities.

Once you are satisfied that your plan is working for the pilot group, you can now involve a second group of children. You may return the original group to regular class work and then start the second group. This would continue until all the children have been introduced to use of activities. A second plan would be to set your pilot group to work on a new set of activities, while you initiated the new group to the original set of tasks. This would eventually lead to the day when your entire class would be actively working at one time.

Time allotted for each activity is to be determined by the ability and interest of the students involved. However, you will find that, in general, this method will require somewhat more time than the normal class period, due to its exploratory nature. As you become more confident and the students more accustomed to the activity approach, you will want to gradually increase the number of periods per week devoted to activities.

Another method of getting in to activities gradually would be through the use of a mathematics corner. Each day a new group may work on a set of activities, while the teacher teaches the rest of the class in her usual way. After the entire class has completed the set of activities, ample time should be taken to display and discuss the students' work. As in the previous method, we want to work toward total class involvement. As the students adjust and show more responsibility toward their work, a second area may be established and more children can participate.

At this point a brief mention will be made for those teachers who wish to get a faster start. The quickest way is to involve the entire class in some degree of active learning. CAUTION! This does not mean you immediately disregard your previous methods of teaching. To do so can cause complications, for both the teacher and students. We do suggest that the class be grouped at work areas to facilitate discussion. Grouping for these activities can be very informal and need not be along ability lines.

A painless way to involve the entire class in activities is to set aside one period a week for activity centers. It is your choice as to the day, but select a day when the children will be most agreeable to something new, perhaps a Friday. The class is then divided into the small groups and presented a task that involves active exploration. You may take two directions here: (1) give each group a different problem or (2) present the entire class the same problem. In the beginning the same problem will prove the easiest. Regardless of your strategy, be sure the problem possesses a good open-ended question.

Selecting Objectives.

Before we discuss ways of selecting objectives, you must be aware that there is no hard and fast rule as to what is appropriate for every child. Mathematics educators, classroom teachers, and textbook authors are not always in agreement as to the make-up of a good elementary mathematics program. The problem is further complicated by the known facts about individual differences in the capabilities of children which the teacher in the classroom must take into consideration in adjusting the curriculum to meet their needs. The selection of objectives should not only be flexible but provisions should be made for adapting it to varying environmental conditions and to the needs of the children in local situations. Ideally, the teacher should assess the needs of her children and try to arrange learning experiences in view of the capability and interests of the individuals. This may all be interpreted to mean the following. Just because a book, curriculum guide, etc. states that a particular objective is to be taught a certain grade, that objective may not be appropriate for your class. Keeping the above caution in mind, what follows should be thought of as suggestive and should be adapted to your situations. Sources such as a text or curriculum guide are valuable, but they should be employed with good judgment.

Basically, you have four sources available to help you select objectives for your students: (1) scope and sequence charts, (2) lists of objectives from publishers, (3) state or local guides, and (4) your textbook. An effective means of employing these is as follows:

Scope and sequence charts. These will aid in giving you a quick, overall picture of what the objectives are for the entire elementary school mathematics program. Such information will assist you in making judgments as to what prerequisite knowledge you should have acquired. Also, it provides a basis for preassessment of students. These charts are available in a number of teacher manuals; if not, they may be obtained from the textbook publishers.

Cumulative list of objectives. Several publishers have made available booklets listing all behavioral objectives for grades K-8. These lists are organized by grade-level and topic. Such lists can be very valuable in laying out tentative plans for an extended period of time. This also provides an excellent guide to the sequential development of particular mathematical concepts.

State and local curriculum guides. Some guides are similar to the cumulative lists of objectives while some provide many teaching suggestions. The state and local guides should be more applicable to your situation since they are intended for your area.

Textbooks. Once you have utilized the above sources, you should have a general picture of what should be taught in your classroom. Now the time has come to use your textbook. The best use that can be made of a text is in daily teaching. The Teacher's Edition can provide suggestions for day-by-day activities that can be followed to attain the objectives you have selected. Some textbooks now come with behavioral or performance objectives in the Teacher's Edition and the Pupils' Text. These specific and sequenced objectives are a good guide to a student's progress through the curriculum.

Assigning Activities

When assigning the children activities, you will need to keep in mind at least two fundamental ideas: (1) the degree of "mismatch" of the activity and (2) the learning-instructional stage at which the child is presently functioning. If either of these ideas are left out of the section process, it can only end in a lack of learning on the child's part and frustration for the teacher.

Determining the mismatch. When a child or group of children are being prepared to work on a new concept, it will be necessary to assess the child's knowledge of the topic. This can be done either formally or informally to determine the degree of familiarity the children have with topics. Informally, you may observe the children during previous units of instruction. A more formal means would be to administer a diagnostic test, either an oral test or a paper and pencil test. In most cases the oral approach will be the easiest and most appropriate. Once your assessment is completed, you can determine the "difficulty level" of the concept to be covered. Without this assessment you may assign activities which are already known by the children or so far in advance of their knowledge that they are incapable of comprehension and will give up.

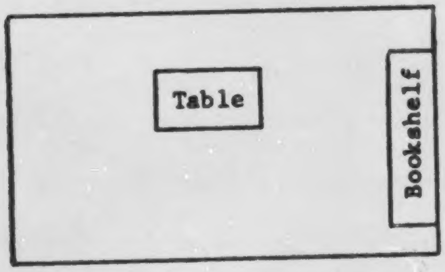
Determining the learning-instructional stage. The learning level (preoperational, transitional, or concrete operational) also plays an important role in what type of activity you assign a child (see section IIIB for selecting activities). There has been an attempt to make all the activities independent of these levels. That is, regardless of level, each child should be capable of completing an activity from his point of reference. However, we must be aware that a child may always select an approach to the task from the "easiest" point of reference. For example, he may always operate at the preoperational level because he feels confident there, yet he may be capable to functioning beyond this point. So when selecting an activity, you not only allow the child to encounter a more advanced arithmetic concept, but you always provide him with the opportunity to function at a learning level just beyond his present mode of operation.

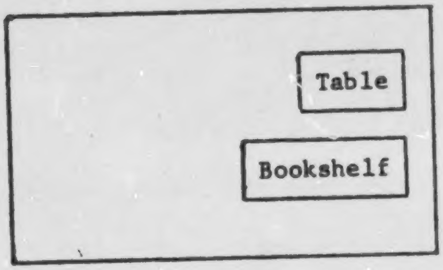
Room Arrangement and Storage of Materials

Special consideration must be given to your classroom arrangement in preparation for activities. The traditional row and column arrangement of desks will prove ineffective with this approach. In an activity-based mathematics program, the emphasis is on individualized learning and a different classroom structure is needed to further this ideal. A teacher should provide surroundings in which an informal atmosphere can prevail. The room should be so arranged as to provide freedom of movement to those students involved and yet shield the remainder of the class from distraction. The lack of physical facilities is most likely one of the biggest reasons teachers hesitate to try an "activity" approach in their classrooms. But let's take a closer look around us. If we use our imagination and a little creativity, we can come up with a room arrangement that will fit out purpose.

If you are going to begin with only a small group of students, the back of the room or a corner is your best bet. It is essential to provide as much separation from the rest of the class as possible. (Distraction can work both ways and you want to give the activities as good a chance to succeed as possible.) Once the location is selected, a flat working area must be provided. This can be a table large enough to accommodate the entire group, individual tables, or a carpeted floor.

You need to set the work area off from the rest of the class in some way. This does not need to be a total isolation, but should at least give the feeling of being something different from the rest of the room. If you have some free standing dividers, such as chalk boards, bulletin boards or book shelves, you should use them. Otherwise, you must improvise. Two arrangements are shown below; one is an easily-created activity area.





Storage space is the next problem we must face. In most cases this will be a premium. One basic requirement for selecting any storage area is that the materials must be readily available to both pupils and teacher. Equipment and material are of little value if they are not accessible at the time they are needed.

In selecting your storage you should keep in mind the following: (1) has the available space been efficiently used; (2) are the units easily moved; (3) do the units serve more than one purpose; and (4) are the units so located and labeled as to promote good housekeeping by the students. The more flexible your stage, the more efficiently the classroom can be organized for activity learning. Your storage units may serve as display areas, room dividers, and even as work space for the children.

Since each of your storage problems will be somewhat unique as far as space is concerned, only general suggestions as to containers and shelves will be made. You will have a variety of small items that will need storage. These can be stored in shoe boxes, cardboard boxes, or rectangular plastic dish pans. If you use the cardboard boxes, you might wish to decorate them with colored paper. In each case the boxes must be clear labeled as to content. Book shelves and areas for large items are usually the biggest problem (not enough money). A good substitute can be built by using concrete blocks and 1" x 12" common pine. If this is to be used, you will want to place them well out of the traffic pattern since they are rather difficult to move.

If you have decided to begin activities with your entire class, your room will need careful planning. For maximum utility, your desk or tables should be arranged in clusters. It has been found advantageous to divide the classroom into sections. This is even more essential when students are working on different topics. Bookcases, portable screens, chalkboards, etc., can be used in this instance. If a sink is available, this area should be partitioned off as an experimental section; if not, another should be prepared for such uses. As your program progresses, there will be desks or tables unoccupied; these should be placed against the wall to provide more floor space for activities. Finally, remember the out-of-doors, weather permitting, can make a good learning center.

Evaluation

Evaluation, while not a unique problem, does pose some special challenges when we attempt to apply "old" axioms to the activity-based program. Traditionally, student evaluation has relied heavily on computational skill, placing a premium on speed and accuracy. At best this is questionable, and it is certainly not appropriate for situations where an activity approach has been employed.

Since all children will not be at the same learning level, we cannot expect all of them to have arrived at the same terminal behavior. Thus, we cannot merely look at outcome, but must attempt to investigate the actual process of learning. This logically suggests the utilization of some form of performance test.

A performance test can be thought of as both a means of evaluation and an extension of the learning process. Since the students are actively involved in learning mathematics, it should follow that active methods should be used for evaluation. A performance test requires active involvement on the part of the student. For example, the student may be presented an activity that requires him to respond to certain questions, the answer to which may be derived from performing the activity. In some instances concrete materials may be made available to the student to solve the problem. However, the student decides what materials he wishes to use, if any. For example, a student may be asked to solve a two-digit addition problem with regrouping. The teacher can provide relevant materials (multibase blocks, pocket charts, etc.) for his use if he thinks it's necessary.

A performance test does not just ask for answers to old questions or previously learned knowledge. It may require a student to discover a new relationship through manipulation, experimentation, measurement, drawing on investigating new patterns. A performance test can determine a child's ability to discover patterns, collect and organize data, and assess his competence in problem solving.

The performance test is normally administered in one of two ways. (1) Each question or set of questions can be located at testing stations and the students may proceed from station. (2) The performance test may be included in regular laboratory lessons, thus providing an ongoing evaluation and diagnoses of a child's progress. Regardless of the method selected, you must be able to allow ample time for the students to complete his activity. Again we are not merely interested in his ability to regurgitate prescribed facts.

You must keep in mind that all of the children will not be ready for a test at the same time. Therefore, performance tests will, in all likelihood, be set up for individuals or small groups. Also, since the students are required to complete certain prerequisite activities before testing, there will be some students taking the test far in advance of others. In addition, the individual nature of the activities will make it possible that all students will not be evaluated with the same activity

One of the disadvantages of using performance testing is reporting and recording the results. This drawback is far outweighed by its superiority in assessing student growth in an activity centered situation. A single letter grade would prove to be grossly inadequate as an indicator of a child's progress.

A brief annotated report should be completed for each activity a child attempts. There are many different forms that such reports can take, but they should contain the following minimum information:

1. Name of student
2. Activity
3. Time spent on activity
4. Student comments
5. Teacher comments

The teacher comments should be a result of observation. However, there will be times when it will be impossible for a teacher to observe every child while he is involved in activity. In this instance, a brief discussion with the child regarding the specific activity will suffice. Such a discussion should be an informal exchange of ideas and questions by both the teacher and student. In no way should this discussion become a one-way questioning session by the teacher. As in the case of the observational reports, any comments regarding the child should be recorded immediately on his card. If necessary you may elaborate upon these notes at a later time, but it is essential that you record your impressions while they are fresh in your mind.

A cumulative file should be kept on each child. This will be a combination of both the child's regular classwork and his progress on his individual activity program. Such a longitudinal record can be inavluable in planning day-to-day activities and getting a clear overall picture of a child's strengths and weaknesses. From this the teacher may plan the appropriate follow-up activities for each child.

In most cases you are required to report a child's progress through a letter grade. This will not always serve to communicate the child's actual growth over a specific period of time. It is suggested that letter grades be supplemented by a brief anecdotal or achievement report. This need only include a statement of the work completed by a child and comments on his strengths and difficulty areas.

Chapter V

Objectives and Activities

Sample Activity Sets: Kindergarten

Matching Relations

OBJECTIVES: To correctly use the relations "more than," "fewer than,' and "same number as" in reference to sets of physical objects.

a. Given two collections of objects, the child can pair the objects one-to-one.

b. Given two collections of objects, the child can compare the collection by pairing the objects one-to-one.

c. After comparing two sets of objects, the child can orally state the relation(s) which hold.

d. Given a collection of objects, the child can select, from another collection, as many objects, fewer objects, more objects, one more object, and one fewer object by pairing the objects.

e. Given three collections of objects, A, B, and C, all of different number, the child can order the collections by using the relations more than, or fewer then.

f. Given three collections of physical objects, A, B, and C, the child can infer the relation between A and C by knowing the relations which hold between A and B and between B and C.

MATERIALS: An ample supply of several different kinds of counting objects should be available. Such objects might include popsicle sticks, counting blocks in different colors, bottle caps, etc. Several containers for these objects will be handy to have.

SETTING: A small group, six or eight children, seated around a table is ideal. You may use the whole class but this is usually less effective.

PREREQUISITES: None

MAP PRINCIPLES: Self regulation is a factor in several activities where the children select the size of the sets they work with. Moreover, various games may be made accessible for free selection time where the children pick their own activities. As the children progress through these activities, they move into content topics and cognitive tasks which challenge them, creating desirable mismatch. In each of the following, the children are participants in an active learning situation.

ACTIVITIES:

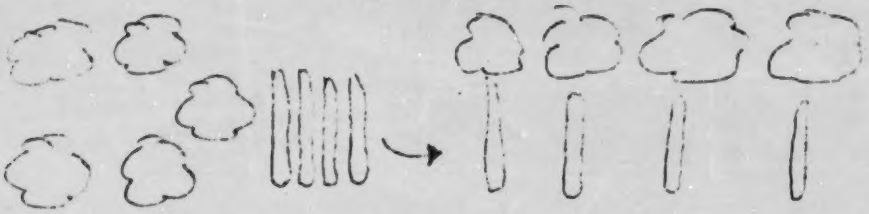
Activity 1 (Exploratory Level)

The purpose of this activity is to have the children form pairs. Have four or five children place their shoes in a box and shake it up. Then have one of the children sort the shoes into pairs. Repeat.

Substitute "Logic Blocks" or "Property Blocks" for shoes. Put the large blocks from two sets of Logic Blocks into the box and shake it up. Have a group of children participate in the sorting activity. Either of these games may be made available to the children on a "free selection" basis.

Activity 2 (Exploratory Level)

Place in small paper bags a variety of different counting objects, exactly two kinds of objects in each bag. For example, in one bag might be bottle caps and popsicle sticks. Each child selects a bag and empties it on the table in front of him. He then sorts the objects into the two kinds and, following that, pairs off the objects. When he has done that, he replaces the objects in the bag and trades bags with

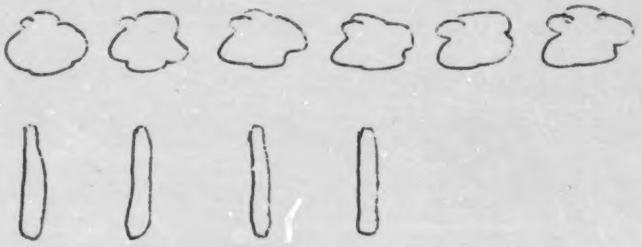


another child. Some bags may have the same number of objects and some different numbers.

Other appropriate activities may be found in the MAP Manual "Matching," Activity Sequence II, pages 10 to 15, and in Silver Burdett Mathematics (1973) "Beginners," Lessons 5, 6, 7, 8, and 9, pages 6 to 9.

Activity 3 (Abstraction and Representation Level)

As in Activity 2, make up some bags with two kinds of counting objects in each bag. For this activity, make sure that the number of each kind of object is small (fewer than ten) and that there is a different number of each kind. The child is to pair the two different kinds of objects (as in Activities 1 and 2) and then tell which set has more. For example, if his bag contained six bottle caps and four popsicle sticks, the child would pair as on the right and state,



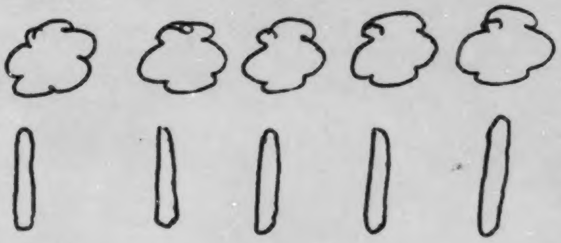
"There are more bottle caps." Ask him how he knows this. If the child dow not know that there are more bottle caps or if he does not know why, tell him. "There are more bottle caps because when you pair them off you have bottle caps left over."

Activity 4 (Abstraction and Representation Level)

As in previous activities, have bags of two kinds of objects available. In this case, have the same number of each kind of object in the bag. A child picks a bag and pairs off the objects and states that the two sets have the same number of objects. If he can tell you that the two sets have the same number, ask him how he knows this. If he cannot state that or explain why the two sets have the same number of objects, then you explain it to him. "These two sets have the same number of elements because, when you pair them off, there are none left over."

Activity 5 (Abstraction and Representation Level)

As before, use bags with different numbers of each of two kinds of objects. When the child has paired the sets, ask him which set has fewer (less) things in it. "Are there fewer bottle caps or fewer caps or fewer popsicle sticks?" If he can tell you, then ask him how he knows. If he cannot explain, you tell him there are fewer sticks because when we pair them there are not enough sticks for each bottle cap.



Activity 6 (Abstraction and Representation Level),

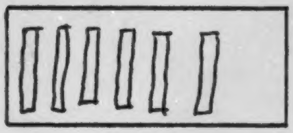
Prepare a box of bottle caps and a box of sticks. Let each child take sticks in one hand and bottle caps in the other, taking as many as he wishes of each. Then he may go back to the table and compare as in Activities 4, 5, and 6.

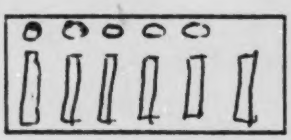
Appropriate activities similar to Activities 3, 4, 5, and 6 may be found in the MAP Manuals "Matching," Activity Sequence III, pages 16 to

23, and "Relations" Pages 2 to 7. In Silver Burdett Mathematics (1973) "Beginners," Lessons 10 to 12, pages 10 to 14.

Activity 7 (Abstraction and Representation Level)

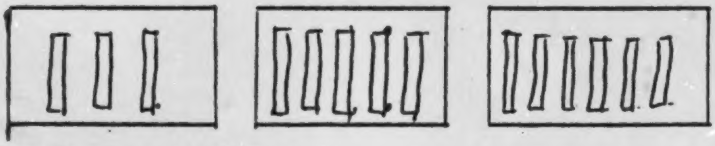
Make 10 or 12 cards by gluing popsicle sticks onto construction paper. Put at least one and no more than nine on each card. Illustrated here is a six stick card. Place the cards and an ample supply of bottle caps in the middle of the table. Now instruct the children to make sets of bottle caps which have (a) the same number, (b) more, (c) fewer, (d) one more and (3) one fewer. The card on the right shows the "one fewer" relationship. If a child cannot do this, help him by example and explanation as in Activities 3, 4, and 5.


Card


Shows "one fewer than."

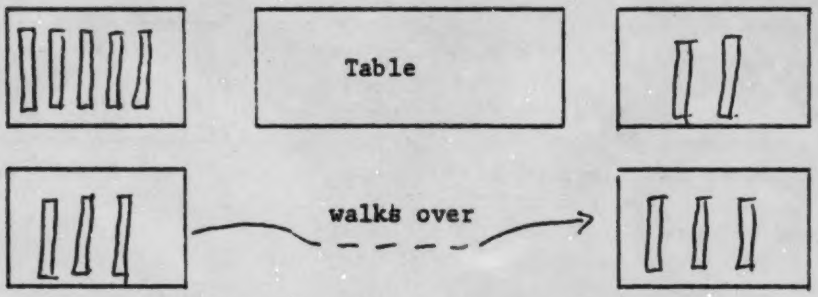
Activity 8 (Abstraction and Representation Level)

Use the cards from Activity 7, and possibly making 15 or so more of the same sort. Have each child select three cards (which must contain three different numbers of sticks) and arrange them in order from smallest number on the left to largest number on the right. Point to the cards two at a time and ask which card has more and which has fewer.



Activity 9 (Abstraction and Representation Level)

Play a game called "Who's Got More?" using the cards from Activities 7 and 8. Three cards with three different numbers are selected by the teacher (or an aide) and each card with the smallest number to stand at one end of the table and the child with the largest number to stand at the other end. The child with the in-between number can move between them. The "between" child compares his set with one of the others and states the relation. Then he goes to the other end and compares cards and states the relation.



"Sam has more than I have."

"I have more than Joe has."

Then the teacher asks, "Who's Got More?", meaning of course, for the two children at the ends of the table to compare their cards, even though they have never been brought directly together.

TEACHER NOTES: Preoperational children should work on Activities 1 and 2 for a considerable time before going on. Preoperational children may participate in Activities through 7, although one would not expect as much comprehension of the underlying principle as with concrete operational children.

Transitional children may work on all nine of the activities presented here. One would expect only occasional success in Activities 8 and 9. Concrete operational children, and not many children in kindergarten are in this stage, would be quite regularly successful in all nine activities.

Numeration

OBJECTIVES: To correctly use spoken and written number names from 0 to 10.

a. The child can say in order the number names from 0 to 10.*

b. Given a set of objects, the child can count the number of objects in the set of objects (0 to 10) and state the number in the set.

c. The child can read and write the numerals 0 to 9.

d. Given a number (1 to 9), in (i) verbal or (ii) written form, the child can form a set containing the given number of objects.

e. Given two sets in one-to-one correspondence, the number of elements in one of the sets, the child can state the number of elements in the other set.

MATERIALS: An ample supply of several different kinds of counting objects will be needed for each child, a supply of 3 x 5 cards and crayons or magic markers.

SETTING: For most of these activities, a small group, six to eight children, will be most effective. These activities can, however, be done in a larger group.

PREREQUISITES: The children starting these numeration objectives should have experience with objectives a to e in the matching relations, kindergarten set. They need not have achieved mastery of the matching objectives.

MAP PRINCIPLES: In many cases the children are self-regulating when they select sets to count. They can control the size of the set they attempt to count, to some extent. They experience mismatch as they attempt to count larger sets, as they learn to deal with the written as well as spoken number names and as they attempt to determine the number in an unseen set from the number in a matching visible set. In this instructional setting, the children are actively dealing with the materials of instruction.

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*No activities are included for this objective.

ACTIVITIES:

Activity 1 (Exploratory Level)

For this activity it is assumed that the children can count in rote fashion to at least ten. Place a set of objects such as books in the center of the table. Teach the children to count the books by having the children count in unison as you touch each book. When you have touched each book ask, "How many books are there?" Many children who seem to be counting do not know that when they finish counting, the number in the set is the last number named. If this occurs, say, "We counted to seven, so there are seven books here." Continue in this way using sets with different numbers of elements.

Activity 2 (Exploratory Level)

As before, place a set of large objects on the table and count them. Ask a child how many there are and encourage him to count them and state the number. When it has been established how many there are, move the objects around. If they are bunched, spread them out and vice versa or build a tower with the objects. When the objects have been rearranged, count them again. Call to the children's attention that when you counted them twice, you got the same answer both times. "How many were there before?...and how many are there now?"

Activity 3 (Exploratory Level)

Encourage the children to select a set of objects at some place in the room. One child goes to the objects and touches them one at a time while the other children count. Then one or more children are asked, "How many ______ are there?" and they respond.

Activity 5 (Exploratory Level)

The children each take a hand full (not exceeding nine) of counting objects and count them. The children form "counting teams" in which one child will point to each object in turn and the other child will say the number names. These counting teams may count various sets around the room and report back to the group the numbers of objects they have counted.

Activity 5 (Exploratory Level)

Begin this activity by having a child "give you a number" between 0 and 9. You then make a set having that number of objects in it by counting out the given number of objects. Ask the children if they think they can make a set having some number of objects and then say, "You can do it...try to make a set of five." The children respond by counting out into a pile of five whatever counting objects you are using.

Activity 6 (Abstraction and Representation Level)

Teach the children to read and write the numerals. No specific activities will be described here but many may be found in the following sources:

(a) Your basal text series for grade one will contain suggested activities for teaching the children to read and write numerals.

(b) "Beginners," Silver Burdett Mathematics, Teacher's Edition, 1973. Use Lessons 94-106, pages 126-140. Use The Pet Shop Book if available.

Activity 7 (Abstraction and Representation Level)

Duplicate Activity 5 but use a numeral to tell the children how many elements to put in their set.

Activity 8 (Abstraction and Representation Level)

Supply cards (3 x 5) and magic marker or crayon to the children. Encourage them to count the elements in various sets such as fish in fish bowl, books on a small shelf, or turtle could be one, an empty cage 0, etc., and label the sets with a card showing the number.

Activity 9 (Exploratory Level)

In this activity we want to introduce the idea that any two sets in one-to-one correspondence have the same number of elements. For this purpose, one set will be hidden, under a box for example. We will suppose you are using red and blue counters. Place a set of red counters in one-to-one correspondence with a set of blue ones. When this has been done, place the box over the blue counters and ask, "How many blue counters are under the box?" If the children cannot answer, count the red counters (there are five) and say, "The sets of red and blue counters matched, so there must be five blue ones also." Repeat until the children can determine how many counters are hidden without help.

SOURCES OF SUPPLEMENTARY ACTIVITIES:

(1) "Beginners," Teacher's Edition, Silver Burdett Mathematics, 1973. The activities in Lesson 15-22, pages 18 to 30, Lessons 68-73, pages 92 to 97, and Lessons 107-108, pages 140-143 are appropriate.

(2) MAP Manual "Counting" and the MAP student workbook "Counting," William D. McKillip and "Counting at Home," by F. V. Souter.

TEACHER NOTES: The activities which involve the use of spoken names for numbers, counting and telling the number of elements in a set and creating a set having a specified number of elements will be appropriate for preoperational children even though they will not grasp the significance as thoroughly as will transitional children. For children who arehaving difficulty using the spoken forms of the numeration activties, it would be desirable to delay the use of written numerals.

Addition

OBJECTIVE: To find the sum of the numbers of elements in two sets of objects by counting the number of objects in the union of the two sets.

a. Given two sets of objects, the child counts each set and then combines the two sets into one larger set (the union) and counts the objects in the union.

b. Given two sets of objects, the child counts the number in the union by counting one set and then continuing his counting to include the other set.

c. Given two sets which are imagined (not physically present), the child counts to find the number in the union, using concrete models of the two sets if he wishes.

d. Given a social or physical situation in which combining sets is the suggested operation, the child counts the number of objects in the union, using concrete models of the two sets if he wishes.

MATERIALS: An ample supply of several different kinds of counting objects should be available. Such objects might include popsicle sticks, counting blocks in different colors, bottle caps, etc. Paper cups (two for each child) in which to keep sets of objects and two plastic jars with lids will also be needed. Several open containers and string loops to form sets will be needed. Flannel board and felt cut outs can be used in these activities.

SETTING: A small group of six to eight children seated around a table is ideal, but you may use these activities with the whole class if necessary.

MAP PRINCIPLES: Self regulation is a factor in these activities as children select the sizes of sets they will work with; the child may also choose to use counting objects or to simply count to solve problems. The activities build from very simple concrete problems to problems involving larger numbers and making use of some imagination, thus encouraging counting without objects. Most children can experience moderate and desirable mismatch as they move along these dimensions. In all of these situations the children are actively involved in the process, emphasizing the activity mode of teaching.

ACTIVITIES:

Activity 1 (Exploratory Level)

Using popsicle sticks and paper cups, form two small sets, say two sticks in one cup and three in the other. Have the children count these two sets separately several times to emphasize the number of elements in each set. Now say, "I want to put all the sticks in one cup. How many sticks will I have if I put them all in one cup?" Make the transfer of sticks and have several children count the set thus formed. Ask the children to recall "How many sticks were in our small sets? How many sticks in the large set?" Do this several times with different numbers of sticks. Do not exceed the limit of the children's ability to count rationally.

Activity 2 (Exploratory Level)

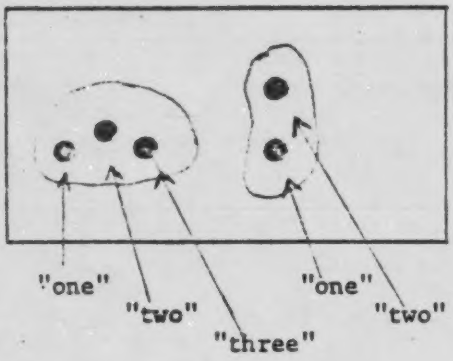
Give each child two paper clips and a supply of small counters. At first, direct their actions by saying, "Put one counter in a cup. In the other cup put two counters." Check that the children have done this correctly and say, "Now, put all the counters in one cup. How many do you have there?" The children count and tell the answer. Do this for many different combinations of numbers. Following this directed activity, encourage the children to explore and report what happens. A child might put three in one cup and four in the other and report, "I put three here and four here. I put them all here and got seven." The children may continue in this way until they show signs of loss of interest.

Activity 3 (Exploratory Level)

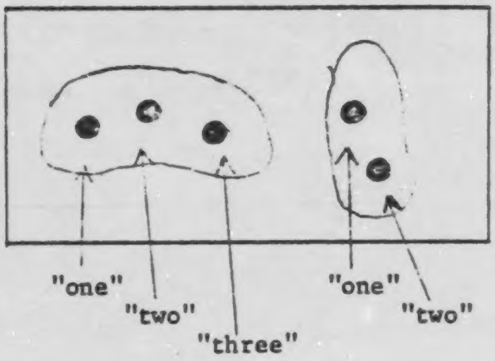
Have the children explore and report as in Activity 2. When a child "reports" a result, have the other children copy his experiment to see if they get the same results. You might say, "Johnny put three and four together and got seven. Let's all try that and see what we get." Encourage discussion about whether we think that three and four will always give us seven.

Activity 4 (Exploratory Level)

Have the children form and count two sets of objects, as in previous activities. These may be sticks in cups or bottle caps in loops of string, etc. Now, we want to find out how many we have in both sets but we will pretend that the objects can't be moved. How can we find how many in both sets? If no child suggests it, you may show them how to count all the objects starting with one set and then "counting on" to the other.



First Time



Second Time

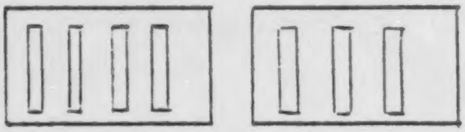
The child may now explore using this new counting technique and report what they get when they count various sets.

Activity 5 (Exploratory Level)

Select sets of objects around the room which are difficult or inconvenient to move (so the sets cannot be combined by physically moving them together). We take for an example two fish in one tank and three in another. Ask the children how many fish "in all" or "all together" or "If we put all the fish in this tank, how many would be here?" Encourage the use of the "counting on" procedure to answer these questions. Let children select other sets around the room and use this counting procedure to determine how many objects are in pairs of these sets.

Activity 6 (Exploratory Level)

Use the cards with popsicle sticks glued to them which were prepared for Matching Activity 7, Kindergarten, to illustrate combinations of sets. Select two cards and count the sticks on each card. When this has been determine, ask, "How many sticks are on both cards?" or "How many



sticks are there all together?" When you have done several examples and the children are using the "counting on" technique efficiently, they may select their own pairs of cards, count, and report what they have found. The same may be done with pictures or dots on cards.

Activity 7 (Abstraction and Representation Level)

Place three counting objects in one plastic jar and two in another. Put the lids on these jars. Rattle them mysteriously and tell the children, "I have three things in this jar (rattle it) and two in this one (rattle). But I can't tell how many things I have. Can you help me?" The game is to tell how many without taking the lids off the jars. One way to do this is to count, "One, two, three...four, five." Some children may be able to do this, but more will not. These children should make sets which match the sets hidden in the jars. For this problem, the children would make sets of three and two and put them together, just as they would do if they could open the jars. When an answer has been found this way, open the jars, combine the sets and count. Do many examples in this way.

Activity 8 (Abstraction and Representation Level)

In this activity the children will be called on to use imagination to a greater extent than formerly. Begin by asking questions like, "Suppose there are three cars in a row. One more car comes up. How many cars?" Let the children answer. Some will be able to count to find the answer but others may need to use counters and pretend they are cars. For some problems you might draw the rows of cars on the board. The children may then enjoy telling similar stories in which sets are joined. Try to emphasize counting "in your head" as a way to find the answer but always let a child use objects if he needs them.

Activity 9 (Abstraction and Representation Level)

In this activity you will describe social situations in which sets are joined. The children are to find the number of elements in the union by counting in their heads or by counting objects. Sample Stories:

"John and Bill are coloring. John has three crayons and Bill has two. How many do they have altogether?" You might act this one out and let the children tell why it would be a good idea for the boys to share.

"Alice and Carol went to the store. Carol got one candy and Alice got two. How many did they have together?" Again, act it out if they need help.

SOURCES OF SUPPLEMENTARY ACTIVITIES:

(1) "Beginners," Teacher's Edition, Silver Burdett Mathematics, 1973. Lessons 38 to 47, pages 53-64.

(2) MAP Manual "Operations," Ralph De Vane and Martin Johnson, R and D Center in Early Educational Stimulation, University of Georgia, 1970, mimeographed.

TEACHERS NOTES: Transitional children will for the most part be able to work through all of these activities, as will concrete operational children, although it is unlikely that you will have any fully concrete operational children in your kindergarten class. Preoperational children may be expected to experience difficulty in working by counting in the absence of concrete objects. Do not force these children to proceed to a level at which mismatch becomes frustrating.

Geometry

OBJECTIVE: The child learns to use the concepts of "open" and "closed" in reference to geometric figures.

a. Given a set of figures, the child can classify them as "open" or "closed" figures."

b. Given a set of boxes, jars, etc., the child can correctly identify those which are open and those which are closed.

c. If a geometric figure is closed, the child can identify the "inside" and the "outside" of the figure.

d. The child can relate himself to other objects, using "inside" and "outside" to correctly describe this relation.

MATERIALS: A variety of boxes with lids and plastic jars or bottles which have lids, several pieces of soft rope about 25 feet long, yarn in several lengths and two colors, paper and crayons, glue.

SETTING: A small group of six or eight children will be best for these activities. The activities involve moving around as well as seat work so an open space is essential.

PREREQUISITES: None.

MAP PRINCIPLES: In each of the activities described below the children are active both physically in manipulating objects in their environment and verbally in describing what they are doing. Mismatch occurs as the children explore ideas which are new to them. Mismatch may be especially apparent in Activity 8 in which two criteria are introduced. The children are self regulating when they create their own games and experiences during free exploration and in the complexity of open and closed designs they make in Activity 7.

ACTIVITIES:

Activity 1 (Exploratory Level)

Leave the boxes and jars, rope and string and other materials out for several days. Permit the children to use this material in any way they wish during a free selection of activities period. When the children are playing circle games, tie the ends of the rope together and put that on the floor for the circle.

Activity 2 (Exploratory Level)

Use the words, open, closed, inside and outside in the normal course of class activities for the day. Encourage the children to use these words. Many circumstances will give opportunities for these words to be used: Put toys away inside the toy box and close the box; open the door and go outside to play; open the drawer, put a pencil inside it, and close the drawer. All of these uses of the words open, closed, inside and outside correspond more or less to the technical usage, although not exactly in every case.

Activity 3 (Abstraction and Representation Level)

Have a group of children play "Simon Says" in a circle. Each child holds the rope at about chest height, forming a circle. One child says, "Simon Says open the circle and let me inside." If he forgets to say Simon Says or to use the words open and inside, the circle does not open for him. When he is inside, he asks to get outside in the same way. Once one child has gotten in and out again, he holds the rope and another child takes his place going in and out.

Activity 4 (Abstraction and Representation Level)

With the group seated around the table, provide for each child a small toy and a box with a lid, or a jar with a lid. Have the toys inside the boxes to start. Each child can peek into his box to see what his toy is. Then each child, in turn, says, "I have a toy inside this box." Then he describes it and other children try to guess what it is. At the end each child opens his box and takes his toy out of the box. The teacher now has a "mystery box" in front of her. She asks, "Is there a toy in this mystery box? I can't open this box. How can we find out?" The children may suggest things to do to tell whether anything is inside the mystery box. They may "heft" it for weight and shake it to hear the rattle. They then may guess what's in it. Teacher then shows them. The activity ends with each child in turn opening his box, putting the toy inside the box and closing the box, telling at each step what they are doing.

Activity 5 (Abstraction and Representation Level)

On the floor place a rope knotted to form a closed curve. Let each child in turn walk around the rope, stand outside and then step inside. Each child should be encouraged to verbalize what he is doing. When all children have done that, we play a slightly different game. Ask each child in turn to place a book or block inside the curve and to place one outside the curve. When several objects have accumulated inside and outside the curve, ask the children "Where is the _____, inside or outside our rope?" Then ask each child to name all the objects inside and outside the curve.

Activity 6 (Abstraction and Representation Level)

With the children sitting at the table, pass out the mimeographed sheet shown on the next page. Let the children examine the curves shown.

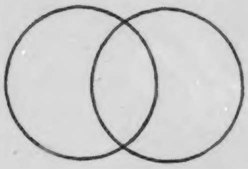
Ask, "Which ones are closed? Which ones are open?" Explain that we can imagine these to be dog fences. If the dog can get out, the fence is open. If the dog cannot get out, the fence is closed. Using crayons the children may now color each curve black. Starting at the x, each child may color red all the space he can without crossing the curve (that is, all the space the dog can run in, if these were fences).

Activity 7 (Abstraction and Representation Level)

Using yarn and glue, the children make a design of open and closed curves on construction paper. Tell them to use two colors of yarn, one color for open curves and one for closed curves. When these are dry, the children may make a bulletin board display of them.

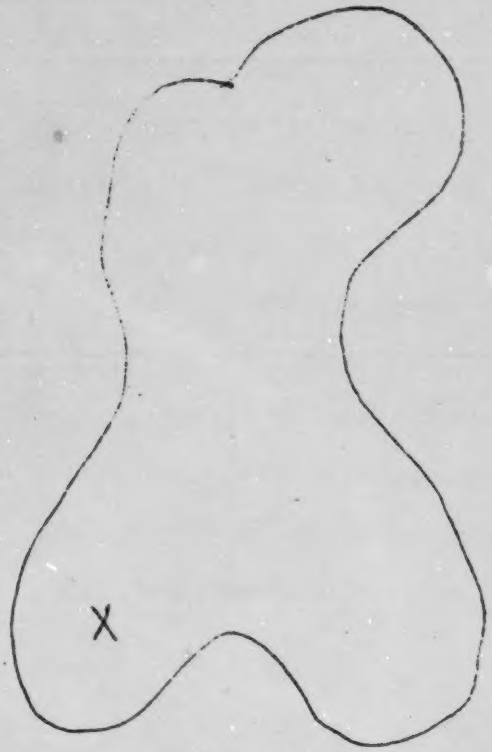
Activity 8 (Abstraction and Representation Level)

Place two ropes as shown on the floor. Name the ropes for the children who placed them, for example, "John's Rope" and "Mary's Rope." Let the children stand in various places and describe their position as "I am inside John's rope and I am outside Mary's rope." Continue with each child in turn taking different positions.



Activity 9 (Abstraction and Representation Level)

With the two ropes placed as in Activity 8, have children put toys in various positions. Then call for them to tell where they have put them, referring to inside and outside of each rope. When several toys have been placed in that way you may call for a child to name all the objects which are "inside John's rope and inside Mary's rope" or "outside John's rope and inside Mary's rope," etc.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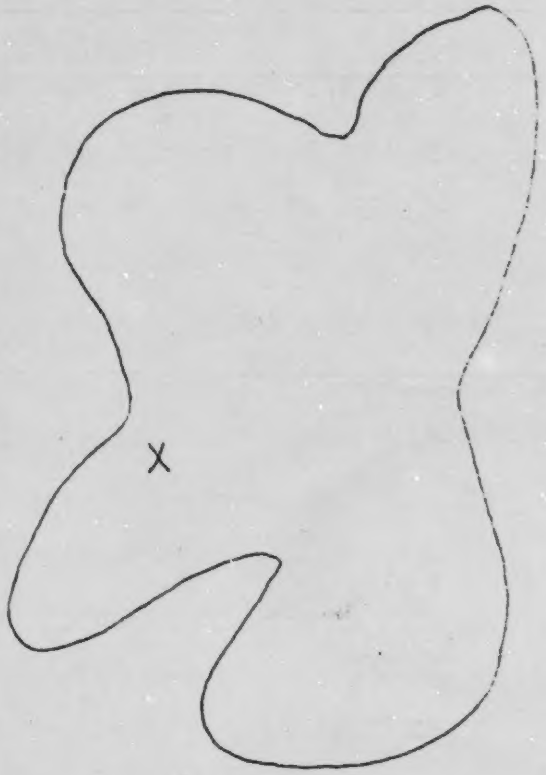
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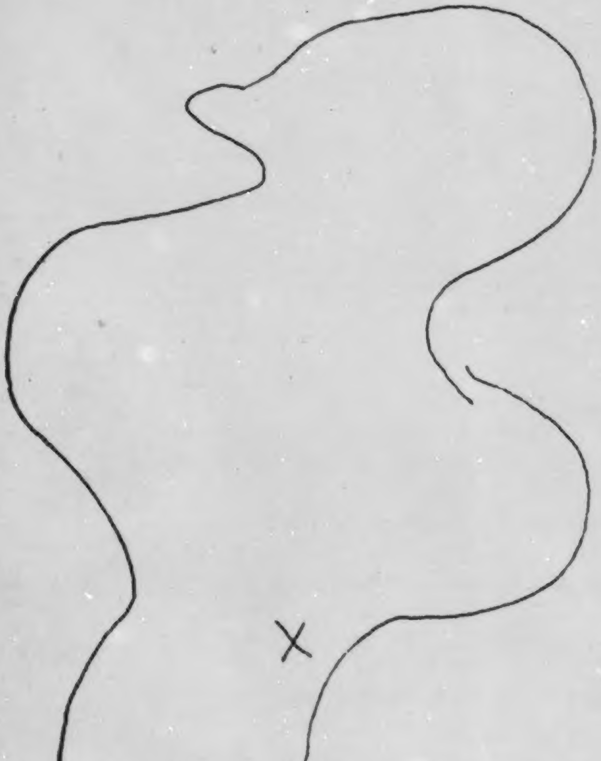
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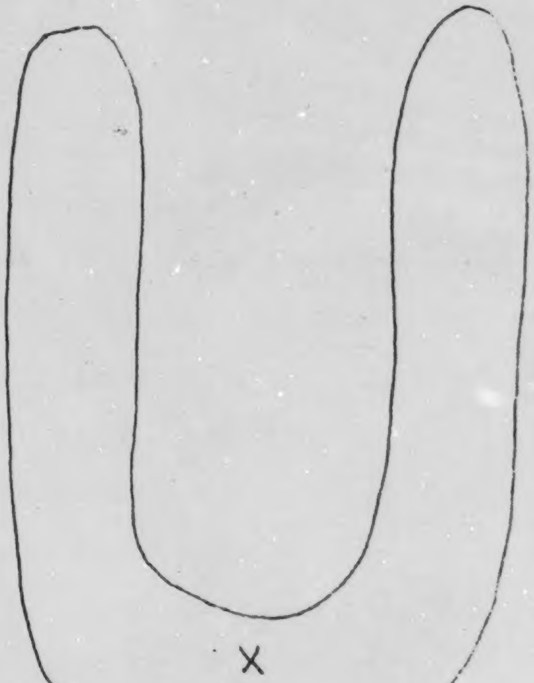
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SOURCES OF SUPPLEMENTARY ACTIVITIES:

(1) "Beginners," Teacher's Edition, Silver Burdett Mathematics, 1973. Activities on pages 2-5 are appropriate for this topic.

(2) MAP Manual "Geometry," Michael L. Mahaffey, R and D Center in Educational Stimulation, University of Georgia, August 1969, mimeographed. Use Activity Sequence IV, pages 32 to 41.

(3) MAP Manual, "Supplementary Mathematics Materials," W. McKillip and F. Souter. Use the material on sorting logic blocks to reinforce these concepts.

TEACHER NOTES: Preoperational children and transitional children will be able to perform most of these activities and will learn the use of the words involved. You may expect that preoperational and transitional children will have some degree of difficulty with Activities 8 and 9. The problem is that two criteria are used to state the position of the object. An object's position may be described as "inside John's rope and outside Mary's rope" or "inside John's rope and inside Mary's rope." It is difficult for preoperational and transitional children to hold two criteria in mind simultaneously. Concrete operational children, and few will be found at the kindergarten level, may do all nine activities with relative ease.

Sample Activity Sets: Grade 1

Matching Relations

OBJECTIVE: The child learns the relations "more than," "fewer than," and "as many as" and "greater than," "less than," and "same number as."

a. Given two collections of physical objects, the child can determine the relation between the two collections by matching.

b. Given that a child determines the relation between two collections by matching, he can count the objects in each collection and order the associated numbers.

c. The child can write and interpret the symbols " > ", " = " and " < ."

MATERIALS: A variety of sets of physical objects. You may find that pairs of sets having obviously related characteristics, e.g., a set of cups and a set of plates, or a set of cowboys and a set of Indians, or a set of red checkers and a set of black checkers, make matchings somewhat easier for children.

SETTING: While teacher directed demonstrations can be held with large groups, each child should be engaged in forming his own matchings and talking about them. This will be easier to do in a small group.

PREREQUISITES: Reasonable mastery of the objectives listed under "Matching Relations" at the kindergarten level. Ability to count to ten.

MAP PRINCIPLES: Mismatch can occur as the children are presented with different collections of objects and changes in directions from the teacher (multiple embodiment). Mathematical variability can provide mismatch as the children progress from exploratory to formalization and representation level activities. Each child should be active in working with his own materials and given the opportunity to regulate the extent of his participation in the various learning experiences.

ACTIVITIES: The following three activities should be considered as exemplars of each type of activity. Others will have to constructed.

Activity 1 (Exploratory Level)

Have the children find partners for a "match-up" of pairs of objects from two different sets. For example, you can have them "match-up" cowboys and horses by placing a cowboy on each horse. Talk about the pairings. Use them to introduce and illustrate the relations "more than," "fewer than," and "as many as" and "greater than," "less than," and "the same number as."

Activity 2 (Abstraction and Representation Level)

Have the children match pairs of sets and state the correct relation. Then have the children count each set and answer questions like: Which of these two numbers is greater? Which of these two numbers is smaller? Is ____ more than or less than ____? How can you tell? The children should be encouraged to answer these questions on the basis of the matchings, i.e., on the fact that there are some objects from one set that do not have partners on the other set.

Activity 3 (Formalization and Interpretation Level)

Have the child fill in the blanks on worksheets with the symbols " > ", " = ", or " < ." Questions similar to the following are appropriate.

7 ____ 9

10 ____ 2

____ < 5

____ = 3

Also include questions similar to these: (you may have to present them verbally)

a. I am more than two but less than five, who could I be?

b. I am more than zero, who could I be?

With your more advanced children, you may want to tackle some difficult questions such as:

a. Suppose I write a big three (3) and a tiny five (5). Can I still write 3 < 5? Why?

b. Donna said, "I am five and Patti is six -- but I'm bigger than Patti -- so sometimes five is more than six." The teacher said, "Wait a minute -- five is always less than six." How do we help Donna?

TEACHER NOTES: In Activity 1 be sure to include both "more" and "fewer" in talking about the same situation. For example, if there are more cowboys than horses, then there are also fewer horses than cowboys. It is easy to always focus attention on "more." Don't forget "fewer." In Activity 1 conservation of the matching relations and conservation of quantity can be encouraged by the teacher's arranging the visual appearance of the sets in various spread-out and bunched formations and asking the child to predict the correct matching relation. Preoperational children should stay with activities of this exploratory and introductory nature.

Mathematical variability can be used to modify Activity 2 by doing this activity with three, four, five or even more sets of objects. Activity 2 can also be modified to have children construct sets containing more, fewer, and as many members as a given set after they have counted the first set.

At the level of Activity 2 (Abstraction and Representation) other activities where the child uses the symbols >, =, and < should be constructed.

Transitional children can engage in both exploratory and abstraction and representation level activities.

Numeration

OBJECTIVE: The child learns numeration through 99.

a. Given a collection of physical objects ≤ 99, the child can partition

b. The child can rationally count sets of physical objects ≤ 99 objects.

c. The child can rationally count by ones from 0 to 99.

d. The child can rationally count by ten from 0 to 99.

e. The child can represent a sets of ten and b ones by the numeral ab.

f. The child can orally state the correct number name for a numeral, for example, "forty-seven" for "47."

g. Given a numeral, the child can interpret the numeral by using physical objects.

h. Given two numerals the child can tell which stands for the larger number and which stands for the smaller number.

MATERIALS: Each child should have an ample supply of counting objects such as popsicle sticks, bottle caps, and rubber bands, plates, jars, or plastic bags in which collections can be kept.

SETTING: A small group of six or eight children seated around a table is ideal. (You may do these activities with the entire class, but they will be less effective.)

PREREQUISITES: Ability to count to 20, recognize and use the numerals 0-9.

MAP PRINCIPLES: The children are self-regulating when they select the appropriate size set they want to count and in the counting procedure they use. Mismatch can occur from the mathematical variability principle as the children are encouraged to count larger sets and to count by tens as the activities given below are modified using the multiple embodiment principle. Each child will be active in selecting and counting sets of objects.

ACTIVITIES:

Activity 1 (Exploratory Level)

Have children count sets by ones and "bundle" each set in a bag, container, or with a rubber band. [If a child counts by ones but makes errors, help him to count by ones correctly by having him initiate correct behavior. Suggest that this child work on small sets of sticks at first.] For sets less than ten, have the children write the numeral telling how many they have. Then have children construct sets of objects having a given (by the teacher) number of members. Have children count sets constructed by their classmates.

Activity 2 (Exploratory Level)

Have children select a set of objects and have them count their sets. Have the children divide each set they counted into bundles (groups) of ten with a group of ones left over (this group of ones should not be bound together but left as individual objects). Introduce the decade names, i.e., twenty, fifty, etc., and practice using them.

Activity 3 (Abstraction and Representation Level)

Have the children count a set of objects and as in 2 above. Have the children say (in their own words) what they have done, e.g., "I have counted cowboys and put them into two sets of ten and have four ones left over." And, "I can put twenty-four cowboys into two sets of ten and one set of four." Continue until children can make both types of responses, i.e., fifty-seven can be bundled as five tens and seven ones and five tens and seven ones make fifty-seven.

Activity 4 (Abstraction and Representation Level)

Repeat the steps of 3 above and then have the child use written symbols like: two tens and four ones (or a tally chart) to write down his results. Pictures of sets of objects should be used to help children make the transition from concrete objects to symbols.

Activity 5 (Abstraction and Representation Level)

Have children do the bundling as in Activity 3 above but have them use the symbol 24 instead of two tens and four ones. Discuss the place value meaning of the numerals and the importance of the order of the numerals. One way to do this is to contrast sets of 42 and 24. The digits are the same but in different places. They should be encouraged to read 24 two ways; as "two tens and four ones," and "twenty-four."

Activity 6 (Formalization and Interpretation Level)

Present children with symbols such as four tens and seven ones, and 63 and have them construct the appropriate collection of objects.

Activity 7 (Formalization and Interpretation Level)

Have children order collections of numerals. Card games where the child with the largest (or smallest) number wins "tricks" can be enjoyable, and instructive, and serve as practice.

TEACHER NOTES: Preoperational children should work on Activities 1 and 2 above. Check to make sure these children are counting correctly and accurately. To encourage conservation, you could have a child count the same set three or four times -- after arranging the objects in various "tight" and "spread out" formations. Encourage the child to use spoken and written symbols in talking about each collection he counts.

Transitional children can be asked what will happen to the number of sticks in each of their collections if these sticks are rearranged. Transitional children can also be asked to draw pictures showing the number of bundles of ten and the ones they have. These children should use the written and spoken symbols and be asked to relate these symbols to the bundles and single sticks (Activities 3-5). Transitional children can also "count" pictorial representations of sticks -- perhaps a picture drawn by another student. Transitional children should wrok Activities 1, 2, 3, and 4 and 5.

Concrete operational children should work all the activities (1-7). They can check one another's work and should be encouraged to resolve any differences between the results of counting the same collection of sticks. In doing Activity 5, concrete operational children can add and subtract groups of tens to, or from, their original set. For example, the teacher could say, "Mary, suppose you have 87 popsicle sticks and I gave you another bunch of ten sticks. How many would you have?" After many such exchanges, the child may abstract the "easy way" to add or subtract tens. This child should be encouraged to formalize this abstraction in his own words. Harder examples can then be used, e.g., "Suppose you had 87 sticks and you gave three bundles of ten to Mary, and then I gave you one more bundle of ten. How many would you have left?"

Addition

OBJECTIVE: The child learns how to find the sum of any two whole numbers a and b where $a + b \leq 18$.

a. Given two whole numbers a and b whose sum is ≤ 18, the child can select sets of a and b objects.

b. Given two collections of objects, the child can find the sum of the numbers of the collections by counting.

c. The child finds the sum of two numbers, a and b, by constructing sets of a and b objects and counting the total collection.

d. The child uses sets of objects to show that $a + b = b + a$.

e. The child uses objects or tally marks to solve equations of form $a + b = \square$.

f. The child can construct (work) number sentences of the form $a + b = c$ to describe the union of two sets of objects.

g. The child begins to memorize the basic 100 addition facts.

MATERIALS: A variety of sets of physical objects suitable for counting. Drawing paper and pencils.

SETTING: Groups of three to six children.

PREREQUISITES: Ability to count to ten. Reasonable mastery of the kindergarten level objectives listed under Numeration, Addition, and Matching Relations.

MAP PRINCIPLES: Mismatch occurs when the activities below are varied using the principles of multiple embodiment and mathematical variability. Each child should be allowed to regulate his participation in the learning activities. The activities call for the child to participate in the construction of new concepts, generalizations, and skills.

ACTIVITIES: The following seven activities are examples of learning experiences at the exploratory, abstraction and representation, and formalization and interpretation levels. Other activities, as well as modifications of those below, will have to be constructed to help children reach the learning objective above.

Activity 1 (Exploratory Level)

Have a group of children around a large pile of objects, e.g., checkers or wooden disks. Call out two numbers ≤ 18. Each child should construct two sets, one set for each number called out. After the children are familiar with this task, make a game out of the activity by letting the first child to correctly construct the two sets be the one to select and call out the next two numbers.

Activity 2 (Exploratory Level)

This activity is a mathematical variation of Activity 1 above. After the children have constructed the two sets (as in 1 above), have them count their total collection.

Activity 3 (Abstraction and Representation Level)

This activity will probably be most effective if it is conducted with a small group (≤ 3) of children. It is a discussion among the teacher and the children about the things they did in Activities 1 and 2 (and other similar activities). Have some counting objects present to use as references during the discussions. Lead the children to verbalize generalizations that describe the addition of two numbers, e.g., "you can add two numbers, like five and eight by first counting out five things, then counting out eight more, and then counting the whole bunch.

Activity 4 (Abstraction and Representation Level)

This activity has children represent their actions in activities like 2 above. Have each child construct two sets of objects (they can also draw sets of objects on paper) and have them place tally marks or a numeral beside each set -- where the numeral or tally marks indicate the number of elements in the set. Next, have the children combine their two sets (or draw a new set which is the union of their original sets), and have them place the correct tally mark or numeral beside this collection. Have them discuss the significance of the symbols.

Activity 5 (Abstraction and Representation Level)

In this activity the teacher (or one member of the group) constructs two sets, counts them, combines the sets, and counts the total collection. The other members of the group write down what happened (or tell the story with numbers) in number sentences like $3 + 4 = 7$.

Activity 6 (Formalization and Interpretation Level)

Present children with number sentences of the form $a + b = \square$. Have them complete and interpret these sentences by constructing physical representations (pictures are acceptable) of the sentences.

Activity 7 (Formalization and Interpretation Level)

Drill or practice using the basic 100 addition facts in contexts where speed and accuracy are important. Any drill or practice game or exercise (e.g., flash cards) is appropriate. Be sure the children realize that they are memorizing these responses.

TEACHER NOTES: Activities 1 and 2 can be used for all children. Modifications can include the children's trying to do the activities with their eyes closed (or blindfolded). Activity 1 can also vary mathematically to incorporate the relations "greater than" or "less than." For example, the teacher could say, "I am thinking of two numbers. One is greater than five but less than seven (pause to give the children time to construct this set), and the other is greater than ten but less than twelve."

Activity 2 can include conservation of number by having the children observe that while each child's total set of objects looks different (since they undoubtedly will be on a variety of arrangements), each child has the same total number of objects.

Activity 3 can be varied mathematically to lead children to form a more "efficient" generalization describing addition -- something like: "You count out the two sets, then instead of recounting the first set, you just tell how many it has and keep on counting the second set." Be sure to hold your discussions (as in Activity 3) a number of times with each child. Make sure the children can use the objects to show what they mean when they describe the addition process. Also include in the discussions the commutative principle: $a + b = b + a$.

Notice how in Activities 4 and 5 the objects and the symbols are playing equally important roles. Be sure to have the children discuss the relationships between the symbols and the objects. Variations of Activity 5 would include the child's selecting the correct number sentence(s) from a small set of number sentences each claiming to represent the actions taken on the objects. Another variation would be the solution of open sentences such as $5 + 3 = \square$, where the child finds the correct number to place in the $\square$. *Caution:* sentences of the $5 + \square = 8$ or $\square + 3 = 8$ are *much* more difficult than those of the form $5 + 3 = \square$. They are called "missing addend" problems and require thought at the level of concrete operations. Missing addend problems are not appropriate for transitional children.

Activities 6 and 7 are for concrete operational children who have mastered objectives a - e above. Drill on the basic addition facts is not recommended for preoperational or transitional children. Drill should be made as enjoyable as possible. Be careful not to overwhelm the children with too many new facts at one time. Start with a small $(n \leq 5)$ subset of the 100 basic facts and build on them. It is imperative that children (when drilling) realize that they are trying to memorize -- they are not to be practicing how to *figure* sums out "real fast."

Geometry

OBJECGIVE: The child learns the names and characteristics of the following geometric shapes: triangle, square, circle, and rectangle.

a. Given an assortment of various sized triangles, squares, circles and rectangles, the child can partition the assortment into sets of triangles, squares, circles and rectangles.

b. The child can correctly name the objects in each of the sets specified in part a.

c. The child can do the activities described in a and b, when he can only handle the shapes but not see them.

d. The child can verbally differentiate between triangles, squares, circles, and rectangles.

e. The child can build models of triangles, squares, and rectangles from sticks, or similar materials.

MATERIALS: A set of Logic Blocks or a set of cardboard geometric figures. (24 figures to include various triangular, squares, circular, and rectangular shapes)

SETTING: Groups of six to ten children seated at a table or on the floor.

PREREQUISITES: Ability to count.

MAP PRINCIPLES: Self regulation will be evident in the strategies employed by the children in the attempts at the partition of the figures into sets. As the child "grows," his strategies will become more to the point.

ACTIVITIES:

Activity 1 (Exploratory Level)

For this activity you will need one set of Logic Blocks or cardboard cut outs mentioned under materials. Have the set of geometric shape in clear view of all of the students. Ask one child to "straighten-out" the pile of geometric shapes. There are several ways he can do this, such as, color, shape, etc. After he has completed the task, ask another student if he knows why the "first" child had arranged them as he did. Discuss his suggestion and then ask if someone could think of another way to arrange the set. After the child has completed the task, have the group discuss the new arrangement. Continue this activity as long as the students can arrive at new "classification" strategies.

Activity 2 (Exploratory Level)

Use the same set of objects used in Activity 1. Begin the activity by placing three triangular shapes in a pile. Tell the children you have a "plan" for what should go in the pile. Ask one of the children to place another one of the figures that would follow your plan (we want another triangle). If he succeeds in the task, ask another child to find another figure that would follow the plan. Continue this until all the triangles are gone. If, at any stage, a child should improperly select a shape, discuss his selection and guide him, through leading questions, to a correct selection. This activity should be continued for each of the four shapes available. As this classification activity continues, there will be ample opportunity to discuss the similarities of the different stacks of shapes. That is, the child should observe that one group are all three sided, one group all four sided, and so on.

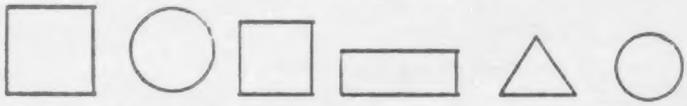
Activity 3 (Exploratory Level)

Place all of the shapes in a pile in front of the children. Ask the children if all the figures are the same. If not, how are they different. There will be several different answers to these questions. The first problem will be getting across what we mean by "different" and "the same." The children will tend to confuse these terms. As the children respond, discuss their answers and then arrange the figures accordingly. For example, the child may say some shapes have three sides and some have four. At this point, have the children separate the shapes into these categories. There will be some figures left; ask about these. Continue until you have discussed all the obvious similarities and differences and have partitioned the figures appropriately.

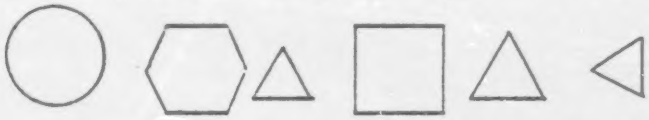
Activities 4 (Abstraction and Representation Level)

As a follow-up to Activities 1 through 3, you may wish to present them with an activity sheet similar to the one given below. Whether or not you give written or verbal instructions will depend upon the reading ability of your students.

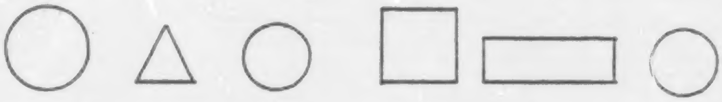
1. Place a (✓) inside all the figures that have four sides.



2. Place a (✓) inside all the figures that have three sides.



3. Place a (✓) inside all the figures that have no sides.



Activity 5 (Abstraction and Representation Level)

After you have completed activities similar to those above, you should use a form of verbal evaluation to check the children's progress. We are mainly interested in their ability to distinguish such things as number of sides, number of vertices, their interpretation of the terms "similar," "like," and "different." If the children perform satisfactorily on the task, you are ready to "attach" the names to the different figures. This endeavor should be no different than teaching any other vocabulary.

Activity 6 (Abstraction and Representation Level)

This activity can be used as a means for teaching vocabulary or as a follow-up to Activities 1 through 4. It should not be used until the children have had ample time to explore the different figures. Such lack of preparation by the children would reduce the value of this activity. Place a set of cardboard shapes (about 12 figures) in a paper sack or box (the children should not be able to see the figures). Have one child hold the bag and direct a second child to reach in the sack and get a triangle. When he brings it out, you should continue as follows:

a. If he is correct, ask, "How did you know that was a triangle?" Discuss his reasoning and make additional comments if necessary.

b. If he is incorrect, ask, "How many sides does a triangle have?" "How many vertices (corners)?" Once he has answered, give him a second chance at the task.

This activity can be continued as long as it is holding the interest of the children.

TEACHER NOTES: All of the above activities are appropriate for the concrete operational child. The preoperational child should be restricted to activities similar to numbers 1 and 2. The transitional child should be able to function satisfactorily in all of the activities. Keep in mind that you are trying to provide the children with an understanding and knowledge of polygons; thus, it is essential that you allow them to explore and discuss all aspects of the shapes.

Sample Activity Sets: Grade 2

Numeration

OBJECTIVE: The child learns numeration through 999.

a. Given a set of physical objects $\leq$ 999, the child can partition the collection into *a* sets of one hundred, *b* sets of ten, and *c* ones where a, b, and c are digits.

b. The child can represent *a* sets of one hundred, *b* sets of ten, and *c* ones by the numeral abc. For example: two sets of one hundred, three sets of ten, and four ones is 234.

c. Given a three digit numeral, the child can interpret the numeral by using physical objects.

d. The child can orally state the correct number name for a three-digit numeral abc. For example, a child can say, "Two hundred sixty-five" for 265.

e. Given two three-digit numerals, the child can tell which stands for the larger number and which numeral stands for the smaller number.

f. The child can rationally count sets of physical objects $\leq$ 999.

g. The child can count by ones, fives, tens, and one hundred from 0 to 999.

MATERIALS: An assortment of *large* sets of objects. By this we mean sets of more than 200, e.g., checkers, tiles, cut out squares or stars, and rubber bands.

SETTING: Due to the large number of objects needed for some activities, a wide open space is helpful. Try sitting a small group on the floor.

PREREQUISITES: Ability to count to one hundred. Reasonable mastery of Grade 1 Numeration objectives.

MAP PRINCIPLES: The children are going to be physically active in counting and organizing physical objects. Mismatch will occur as the children relate physical arrangements to verbal symbols and written symbols, and as the children move through the three levels of activities.

ACTIVITIES:

Activity 1 (Exploratory Level)

Supply the children with an assortment of more than two hundred objects, e.g., bottle caps, wood disks, small cardboard squares, dittoed sheets full of dots or stars. Have the children see how many sets of ten they can make. When they have done this, have them make as many "giant sets" as they can, where a "giant set" is a set of ten sets of ten. They will now have three different sized sets -- a set of ones -- sets of tens -- and sets of ten tens ("giant" sets or some other term the children wish to use for this set).

Activity 2 (Abstraction and Representation Level)

Extend Activity 1 above. Have the children count the number of elements in a "giant" set. Label the "giant" sets as sets of "one hundred." Have them use number names like: "three hundred forty-seven" for collections such as those formed in Activity 1 above.

Activity 3 (Exploratory Level)

Introduce the children to a set of multibase blocks for base ten. Allow the children to play with these materials and then relate them to the sets of ones, tens, and hundreds they have previously constructed. Repeat Activity 1 above using base ten multibase blocks. Note: children can make tens out of ones and hundreds out of tens. Compare the "constructed" tens and hundreds with the preassembled tens and hundreds. They should be the same size.

Activity 4 (Abstraction and Representation Level)

Repeat Activity 2 using the multibase blocks. However, this time introduce the written three-digit symbols for each physical collection the children construct. Practice using the spoken and written names for collections up to 999. Now, using the multiple embodiment principle, construct sets of various objects and label these sets with appropriate three-digit numerals.

Activity 5 (Interpretation and Formalization Level)

Have the children interpret three-digit numerals by giving them a three-digit numeral and asking them to construct the appropriate sized collection. Use a wise variety of objects including the multibase blocks. After the children are able to construct collections from numerals, you can play some games along these lines. Children can vie to be the first in constructing sets. In such competition, the winner can call out the next number and be the judge on the other children's speed and accuracy. Another version of this game would be to have the children perform their constructions while blindfolded -- probably using multibase blocks.

Activity 6 (Formalization and Interpretation Level)

Have the children fill in missing numerals in patterns. Many textbooks have exercise pages along these lines:

297, 298, ___, ___, ___, 302

100, 200, 300, ___, ___, ___

545, 550, 555, 560, ___, ___, ___

Another exercise is to present children with assorted numerals, either on a sheet or on cards, and have them order the numbers from the smallest to the largest. Still another way to have children order numerals is to have them play RANKO (see Teacher Notes). In this game each child draws six or eight cards from a deck. Each card has a numeral from 1-999 on its face. The children must place their cards in a row (face up) in the order they dealt. Then each child takes a turn of either drawing a new card or switching any two of his cards placed in his row. If a new card is drawn, it must replace one "old" card in the child's row. The "old" card is then discarded. The object of the game is to be the first to get all cards in either ascending or descending order.

TEACHER NOTES: For Activity 1 try to have the children form their giant set so that the ten sets of ten are still visible. Talk about the ten sets of ten and count them by tens to one hundred. Then, still using the ten sets of ten as a visual cue, count the giant sets by ones to one hundred. Review the order relations by asking questions such as, "Is forty-seven more or less than seventy-five? How can you tell?" Pose similar questions using numerals, e.g., 48 ___ 84 (when the child inserts <, >, or = in the space). An excellent mathematical variation of the preceding exercise is to "cover up" one or more of the numerals in a situation like 48 ___ 84 (fill in <, >, or =). When one of the digits is covered, the child will see something like: 4_ ___ 84. Note -- the child can correctly justify answers in two-digit "covered up" situations when the ones digit is covered. However, it is not possible to be sure when the tens digit is covered. Talk about why this is the case.

For Activities 2 and 4, extend the "covering up" type questions to three-digit numerals. Here the hundreds digit is the key element. Ask the children why it is possible to answer "for sure" when the ones or tens digit is covered, but not when the hundreds digit is obscurred. Discuss place value here.

The commercial version of RANKO is available from: Midwest Publications Company Incorporated, P. O. Box 307, Birmingham, Michigan 48012. It contains four decks of playing cards. Some of the decks contain more advanced numeration ideas, i.e., exponents and bases other than ten.

As has been noted before, Activities 1-6 are appropriate for children at the stage of concrete operations. Preoperational children should be engaged in the exploratory level activities (1 and 3). Transitional children can profitably engage in Activities 1-4, and can try their hand at Activities 5 and 6.

Addition

OBJECTIVE: The child learns how to solve missing addend problems, i.e., problems of the form $a + \square = b$.

a. Given a collection of n objects and the numbers a and b (where $a + b = n$), the child can partition the collection.

b. From a collection of n objects, the child can form a subset of a objects ($a < n$) and determine how many objects are left.

c. Given a collection of less than n objects, the child can select more objects so that he now has n objects.

d. Given a collection of less than n objects, the child can tell how many more objects he needs in order to have n objects.

e. Given the number sentence $a + \square = b$ (with values for a and b given), the child can construct a model (verbal or physical) for this sentence and find the missing number by counting.

f. The child solves number sentences of the form $a + \square = b$ by subtracting, i.e., $b - a = \square$.

MATERIALS: A variety of sets of small objects. Three x five cards -- some with numerals from 0-20. Paper and pencil and something to use to represent a number line.

SETTING: Groups of three to six children.

PREREQUISITES: Ability to count to 20. Reasonable mastery of addition. Ability to solve sentences such as $15 + 4 = \square$.

MAP PRINCIPLES: Missing addend problems are quite difficult for children. They will tend to change problems like $5 + \square = 9$ into $5 + 9 = \square$, work the latter, and give an answer of 14 for $5 + \square = 9$. Be alert for such behavior. It is a sign that mismatch is present. Variations of the activities, using multiple embodiment and mathematical variability will also cause mismtach. Self regulation should be present in the form of a large number of exploratory activities. Don't rush the child into the abstraction level. Wait until he is adroit in a number of exploratory activities. All of the activities described below call for the children to think about their own actions.

ACTIVITIES: The following activities are examples of learning situations that can be designed by a teacher. Other activities, at all three levels, will have to be constructed and employed to help children solve missing addend problems.

Activity 1 (Exploratory Level)

Have each child select the same number of objects. The children can use any objects they wish. Each child's set may be comprised of different items, but all children should have the same number of items. To begin, keep the number of each child's objects small, e.g., 10. Point out the fact that everyone has the same number of things. Then ask each child to separate his collection into two parts. Leave the objects as they are and ask for number combinations that add up to ten. Write them down so everyone can see the number pairs. Then ask, "Who has this one?" - refer the children to their collections. After doing this a few times, with different sized collections, begins calling out the number combinations first, and let the students separate their collections into two sets.

Activity 2 (Exploratory Level)

This is a mathematical variation of Activity 1. Each child has the same number of objects. Call out a number (less than the number in the children's collection) and have the children "cover up" that many of their objects -- with one hand or a small card. Ask them, how many more do you have?

Activity 3 (Exploratory Level)

This is a mathematical variation of Activity 2. Proceed as in Activity 2 but "cover up" the remaining objects, i.e., have the children select some of their objects and quickly cover up the rest. Then ask them to tell how many are covered up.

Activity 4 (Exploratory Level)

Play "What do we need?". Have a child pick a number (less than 20). Write this number on a card or on the board. Then have another student pick another number -- smaller than the first. Give each child a number of objects equal to the second number chosen (the smaller number). Ask, "What do we need," or "How many more do we need to make this many?" Point to the number written on the card. Let the children use any method they wish to find the answer. Discuss the strategies used to find answers. Encourage verbalization of the student's strategies.

Activity 5 (Abstraction and Representation Level)

Play the "what do I need" game — see Activity 4 -- using large cards. Proceed as follows. Have one child take a card with a number ≤ 20. Another child takes a numeral card with a number smaller than the number on the first card. A third child takes a blank numeral card. A fourth and fifth child take cards with + and = on them. Arrange the children in a row so that their cards look like this:

$$\boxed{7}\quad\boxed{+}\quad\boxed{}\quad\boxed{=}\quad\boxed{12}$$

The student with the blank card asks, "What do I need?" The others help him decide. Discuss strategies. Some students will solve the equation by counting from the first number to the second number, e.g., from 7 to 20 in the above illustration. The strategy you eventually want students to use is subtraction. In the illustration above, ___ = 12 - 7. It may be that students are slow to make this abstraction. That is, see the subtraction strategy. One way to prompt this abstraction would be to use a guided discovery approach. This is discussed below under Teacher Notes.

Activity 6 (Interpretation and Formalization Level)

Have some students play a teacher's role and explain to the group what a sentence like 8 + ___ = 14 means, and how to solve it. In this activity one student is the teacher, while the teacher is the student! The other children observe and judge how well the teacher and student perform their roles.

TEACHER NOTES: We have included a number of exploratory activities in this set. We have done this to emphasize the need to proceed slowly on missing addends. We advise a deliberate approach to this topic since it has a history of being very difficult for children to master.

In Activity 3, the teacher may wish to do the covering up in order to insure that the child does not quickly count the objects and circumvent the intent of the activity.

Activity 4 can employ a number line and a jumping frog or cricket instead of discrete objects.

In doing Activity 5, have the children holding the first and third cards switch places. Observe that the answer -- and the strategy -- stay the same.

One way to guide discovery of the subtraction strategy would be to present children with a series of missing addend equations and to keep a record of the answers. A two column record could be organized as follows. The left column is the original sentences and the right column the solved sentences.

$5 + \square = 11$	$5 + \boxed{6} = 11$
$3 + \square = 7$	$3 + \boxed{4} = 7$
$2 + \square = 10$	$2 + \boxed{8} = 10$
$5 + \square = 9$	$5 + \boxed{4} = 9$
$4 + \square = 10$	$4 + \boxed{6} = 10$

Leading questions such as the following may help children discover the subtraction strategy.

How many numbers do we have to start with? (Point to the left column)

Look at the two numbers in the first one. What are they? (5 and 11) What is the number that goes in the box? (6) How can 11 and 5 give 6?

Look at the two numbers in the second one. What are they? (3 and 7) What is the missing number? (4) Can 7 and 3 give us 4? How?

Multiplication

OBJECTIVE: The child learns the meaning of multiplication.

a. Given an appropriate collection of objects, the child can partition the collection into subcollections, each with the same number.

b. Given the child has partitioned a collection of objects into a subcollections with b per subcollection, he can find the total number of objects by repeated addition.

c. The child can represent a sets with b per set by a x b.

d. Given a product, such as 5 x 3, the child can interpret the product with physical objects and rename the product by repeated addition.

e. The child can count by two's, three's, four's, and five's.

f. The child can use the "one" property for multiplication.

g. The child can use the commutative property in multiplication.

h. The child can use the basic multiplication facts for two numbers, a and b, where $a \times b \leq 18$.

MATERIALS: Several different sets of objects, such as checkers, bottle caps, tiles, etc. Each set should contain 18 objects.

SETTING: Groups of six to ten children seated at a table.

PREREQUISITES: Ability to count and to add single digit numbers.

MAP PRINCIPLES: When introducing multiplication, you must be patient. You are not only introducing a new concept but also new symbols. Allow the children to participate in a large number of exploratory activities before using the abstract symbol.

ACTIVITIES:

Activity 1 (Exploratory Level)

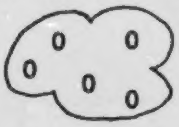
For this activity each child should have access to a set of counters. Have each child place a set of eight counters in front of him. Make sure that they each have eight. Now ask the children to separate their sets into two equal sets and ask "How many counters are there in each set?" "Oh, you have two sets of four, does that make eight counters?" Continue this type of activity until the children can partition any set of 18 or less. At this time, do not write any symbols; we are only concerned with the act of partitioning.

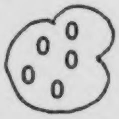
Activity 2 (Exploratory Level)

In Activity 1 the children were given the number of sets they were to obtain and then counted the number of objects in the sets. In this activity you are to do just the opposite. Give each child a set of 12 objects and ask him to place them in piles of four each. When he completes this, ask, "How many sets did you get?" "Good, you have three sets of four; does that make 12 in all?"

Activity 3 (Exploratory Level)

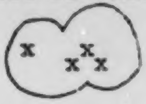
Place two sets of counters on the table in front of the children.

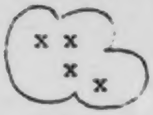


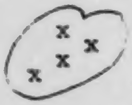


Ask, "How many sets do I have?" (2) "How many counters are there in each set? (5) "Good, I have two sets of five; how many do I have all together?" (10) Continue this activity, giving each child ample time to participate. It is essential that you discuss each example with the children.

The burden of the response should be gradually shifted more and more to the child so that, if confronted by the following, he would respond







like: "There are three sets of four counters and that would make 12 in all."

Activity 4 (Exploratory Level)

In Activity 3, the child had visual cues (counters) from which he could obtain his answer. In this activity, he is expected to perform the task from verbal instructions. Have each child place "five sets of three" in front of him. Ask, "How many counters does he have all together?" "Good, five sets of three are 15." This activity should be used with as many combinations (total number under 18) as is necessary for the child to acquire acceptable performance.

Activity 5 (Abstraction and Representation Level)

Place three sets of five counters where each child can observe them. Ask, "How many sets are there, how many counters are there in each set?" After this response ask for the number of counters in all the sets together. As the children respond, record their response on the board as follows:

"3 sets of 5 are 15"

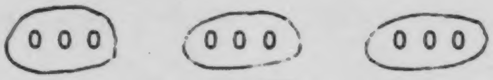
"3 x 5 = 15"

Discuss this new set of symbols with the children and explain that the "x" is read as "times." Tell them this is just another way of saying "three sets of five." After the children have worked through several problems with you, ask if anyone can solve the following:

3 sets of 4 are ☐

3 x 4 = ☐

Be sure to include both sets of symbols at this time. To follow up this activity, a few problems can be given on a work sheet to see if the children can interpret the symbols. Be certain there are accompanying pictures to assist the children in solving the problem.



3 sets of 3 are $\square$

$3 \times 3 = \square$

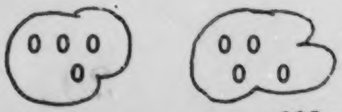
Activity 6 (Abstraction and Representation Level)

Present the children with a problem like $2 \times 4 = \square$ and ask them to solve it using counters. They should arrive at their answer by forming two sets of four. Once they have obtained the answer of eight, ask if there is a way to get the answer without counting. Usually someone will tell you to add 4 + 4; if they don't, tell them. Continue this form of activity until the child can use repeated addition to solve problems of the form $a \times b = \square$.

Activity 7 (Interpretation and Formalization Level)

This activity is to be just like Activity 6, with one exception -- the child should work the problem without using counters. You should encourage the children to solve problems of the form $a \times b = \square$ by using repeated addition.

TEACHER NOTES: In Activities 1-4, the emphasis is on verbal responses and you should not attempt to use written symbols at this time. You should use enough repetitions to enable the child to make the correct interpretations. There are three basic objectives: (1) given a certain number of objects, the child can partition it into specified subsets; (2) given a set like,



the child verbally states: "two sets of four are eight; and (3) asked to find how many counters are in five sets of two, the child forms the proper sets to obtain the solution.

Activities 5-7 are to be used to develop the symbolism necessary to interpret a multiplication problem. The concrete operational child will benefit from all the activities (1-7), while the preoperational child should be restricted to Activities 1-4. The transitional child should be expected to function satisfactorily in activities similar to numbers 1-6.

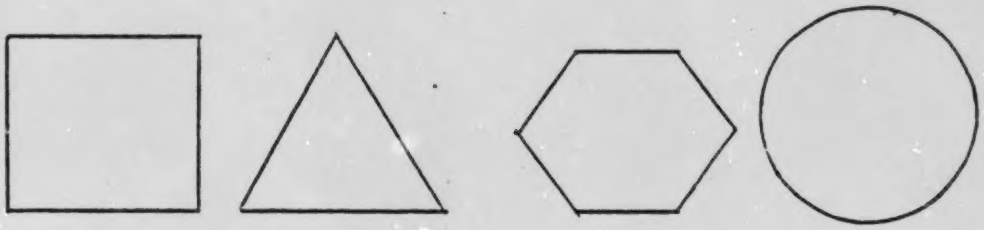
Geometry

OBJECTIVE: The child demonstrates the congruence of two plane figures by "superposition" and describes the congruence as meaning "same shape and size."

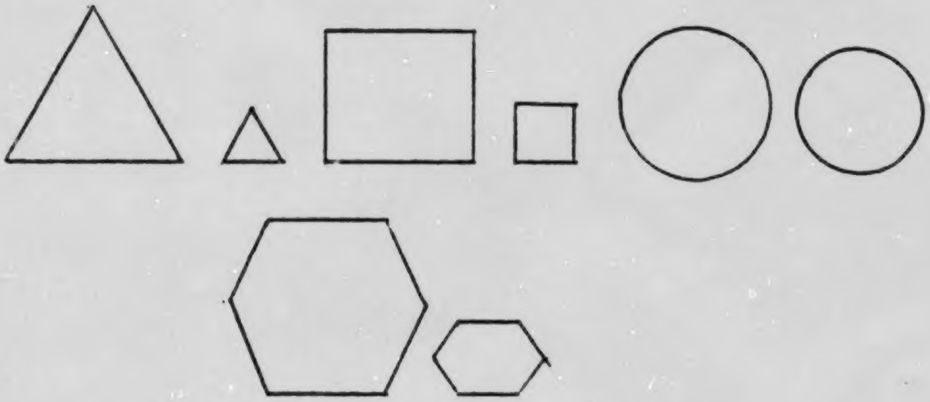
TASK: Given several cardboard models of plane geometric figures, the child will determine which one, if any, is congruent to a given figure. This should be accomplished by placing one figure upon the other.

MATERIALS: Three sets of cardboard cutouts of plane geometric figures. The following illustration should be followed in constructing the figures.

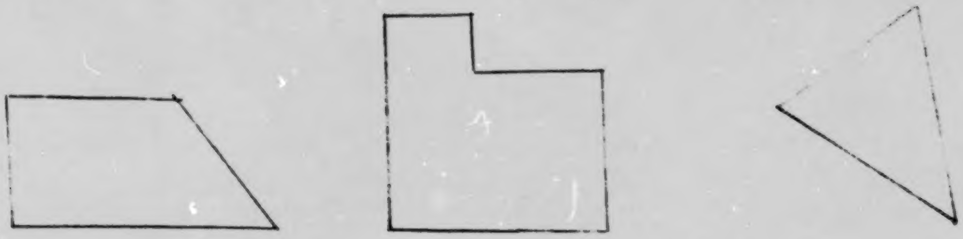
Set A. (The following shapes should be supplemented with other regular figures)



Set B. (The following figures should be supplemented with more similar figures)



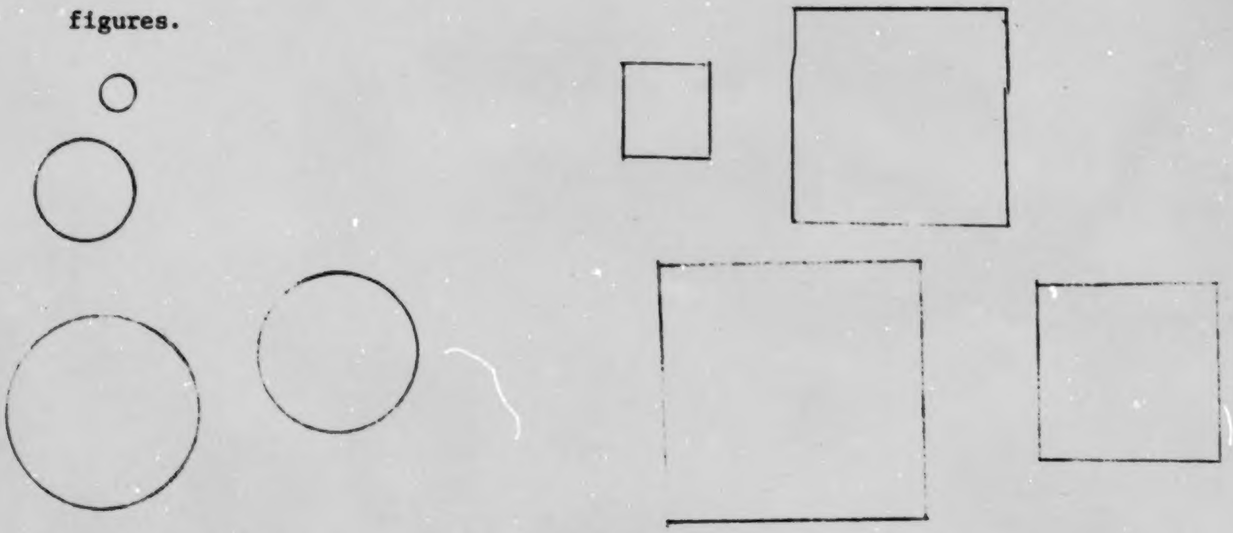
Set C. (These figures should be different colors on both sides)



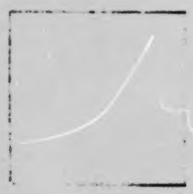
Activity 1:

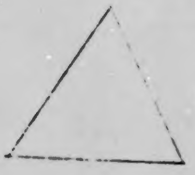
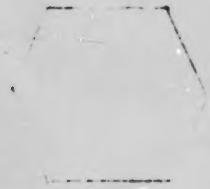
a. Give the child the figures from Set A and Set B. Ask him to find figures from Sets A and B which "fit together exactly." Ask the child to explain why he matched the figures as he did. The explanation should involve concepts of both shape and size.

b. Ask the child to take all the figures in both Sets A and B and sort them on the basis of shape. Ask the child if all of these figures will fit exactly. The answer should focus on differences in the sizes of the figures.



c. Ask one child to sort all the shapes on the basis of size. Ask if the large ones will all fit exactly; ask if the small ones will fit exactly. The answers should focus on differences in shape.



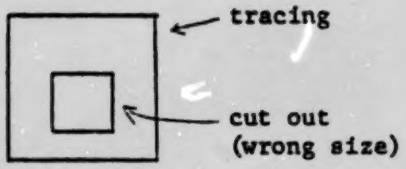



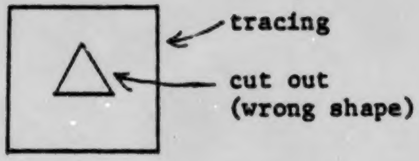
d. Now place a tracing of one of the figures on the table and ask the child if they can find the figure you used to make the drawing. Once a child selects a figure, he must check it by placing it inside the traced lines.

e. Continue until you have used all of the figures.

Activity 2: (Abstraction/Use "Set B" for this Activity)

The children now have the idea that a figure may be placed upon another to see if it covers the tracing esactly; however, each of the figures in Activity 1 were one of a kind. Now we have added another factor and that is, size. Not only must a figure be the same shape as the drawing, but it must be the same size. A child may encounter two different problems. He may find a figure that "fits" inside the tracing, but it's too small or even the wrong shape. Thus, we want him to see that to be congruent it must be the same size and shape. (See below.)





After reviewing Activity 1, proceed with the following:

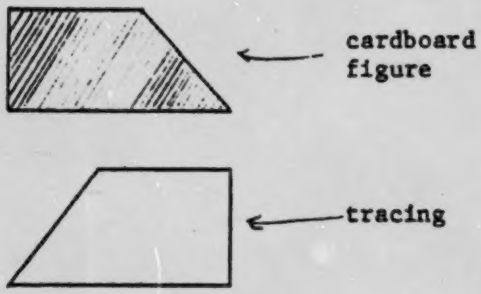
a. Present the group with a tracing of one of the figures shown above. Ask if anyone can find the figure that will "fit" the tracing "exactly."

b. When a figure is selected, allow the child to check his selection by placing it upon the drawing. If it is an "exact" fit, continue until you have used all the figures. If it does not fit, have the children discuss why it doesn't. That is, is it the wrong shape? Is it the wrong size? Allow the children the opportunity to see why it doesn't fit.

c. When the children are capable of making the correct selection (that is, they have made the abstraction), you may continue with the next activity for congruence. It would also be appropriate to have the children start using the term congruence.

Activity 3 (Use "Set C" for this activity)

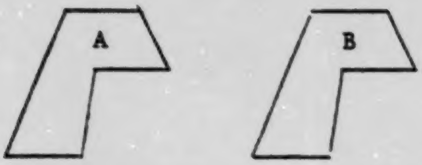
In Activities 1 and 2 above, children should have concluded (abstracted) that two figures "fit exactly" (are congruent) if they are the same shape and size. To check this they needed only to place one figure upon the other. However, in this activity we are going to use irregular figures. This, in some case, will require the student to rotate or flip the figure to get an "exact" fit (see figure below).



As you can see, merely rotating the figure will not make it fit the tracing. It will require the child to "flip" the cardboard figure over to make it fit.

a. Proceed as you did in Activities 1 and 2 above. If a child encounters difficulty, it may be necessary for you to give him a "hint" about "flipping" the figure to get it to fit.

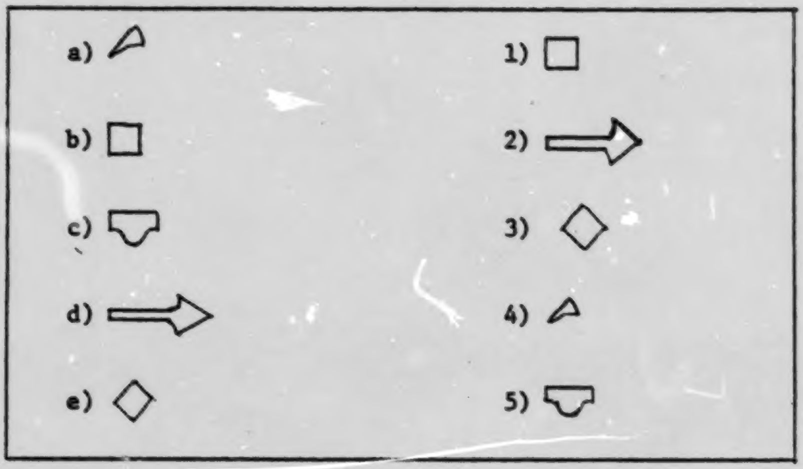
b. The child should be expected to check two figures, by moving one to the other, and then conclude as follows, "Figure A is congruent to Figure B."



Activity 4

Once the child has successfully completed Activities 1-3, he may proceed with the following.

a. Present the child with a worksheet similar to the one shown below.



b. Ask the child if he can find the figures in the second column that are congruent to each of the figures in the first column. It is obvious that he no longer can pick up either of the figures to check and see if it "fits exactly." He will have to find a new way to check his conclusions.

c. You can ask the child it he can think of a way to do it. With some assistance from you, he can see that if he traces one figures and places this over the second one, he can discover if there is a congruent figure.

TEACHER NOTES: Preoperational children will probably work in Activity 1 above. Concentrate on the act of placing the figure within the confines of the tracing. Use each of the figures more than once, if necessary, to make sure the children understand what is meant by "fitting" the tracing.

Transitional children can be expected to work in both Activities 1 and 2 above. You may also begin using the term "congruence" to describe the results of an "exact fit." It would also be expected that these children would begin using the spoken word to describe their action.

Concrete operational children should be involved in each of the above activities. In working with Activities 1, 2, 3, and 4, they can be expected to make a conclusion regarding the congruence of two figures before they are allowed to move the model. Also, they can go one step further and be presented a situation where neither figure is movable and they must employ a form of the transitive property to arrive at a solution.

Sample Activity Sets: Grade 3

Numeration

OBJECTIVE: The child learns numeration through 9999.

a. Given a set of physical objects $\leq$ 9999, the child can partition the collection into a sets of 1000, b sets of 100, c sets of 10, and d ones, where a, b, c, and d are digits.

b. The child can represent a sets of 1000, b sets of 100, c sets of 10, and d ones, by the numeral abcd.

c. Given a numeral, the child can interpret the numeral by using physical objects.

d. Given two numerals, the child can order the number named by the numerals.

e. The child can rationally count sets of physical objects $\leq$ 9999.

f. The child can rationally count by ones, fives, tens, hundreds, and thousands from 0 to 9999.

g. The child can order the numbers 0-9999.

MATERIALS: An assortment of very large sets of objects, i.e., sets with more than 1000 objects! We suggest that you have the children help you construct these sets and that the elements of each set be "packaged" in some way into 10's, 100's, and 1000's. The construction, building, and labeling of the sets is an excellent exploratory activity in itself. Let the children suggest the kinds of objects and the bundling procedures. For example, bottle caps can be placed in plastic sandwich bags and tied with rubber bands. Jelly beans and M & M candies can also be bundled in this way and consumed at a convenient time. Popsicle sticks can be tied together with rubber bands. Small squares of paper or cut out stars can be paper clipped together. The commercial version of multibase blocks can also be used since the large cube is 10 x 10 or 1000 small cubes.

SETTING: Due to the largeness of the physical collections, the floor may be the best place to work.

PREREQUISITES: Ability to count to one thousand. Reasonable mastery of Grade 2 Numeration objectives.

MAP PRINCIPLES: The children should be active in constructing and using the manipulative aids. Mismatch occurs as their numeration includes the thousands place, and as the children change levels of activities.

ACTIVITIES: Following is an example of activities at various levels. Others should be constructed.

Activity 1 (Exploratory Level)

As has been suggested, have the children help construct sets of more than 1000 objects. Children should have responsibility for the packaging of each set into bundles of 10's, 100's, and 1000's and writing the symbols (e.g., ten or 10) on each bundle. Note: a set of 2,345 objects can be bundled in a number of ways, e.g., into two one thousand unit groups, two one hundred unit groups, fourteen groups of ten, and five single units.

Activity 2 (Exploratory Level)

Have the children edit each of the bundles created in Activity 1 above. The rule being: first, make the most 1000 unit groups you can, then make the most 100 unit groups you can, then the most ten unit groups. Make a record of each of the arrangements before editing and after editing. For example, the set in Activity 1 above would originally be recorded as 2-thousands, 2-hundreds, 14-tens, and 5-ones. The editing bundling would give: 2-thousands, 3-hundreds, 4-tens, and 5-ones. Then discuss total number of objects in each case.

Activity 3 (Abstraction and Representation Level)

See Activity # under Numeration Grade 2. Extend it to four-digit numbers.

Activity 4 (Abstraction and Representation Level)

Practice using the spoken and written names for numbers through 9999. Say the name and have the children write the symbols. Give the symbols and have the children write the name. Divide the group into teams and use some type of a game format for this drill.

Activities 5 and 6 (Interpretation and Formalization Level)

Activities 5 and 6 under Numeration Grade 2 can be adopted to include thousands.

TEACHER NOTES: Refer to the Teacher Notes for those activities adopted from Numeration Grade 2. The RANKO game may be especially useful. Children can make their own playing cards for this game.

A mathematical variation of the patterns used in Activity 4 should have the children respond to five-digit situations.

Addition

OBJECT: The child learns to solve addition problems involving addends 99.

a. Given two collections of objects, each consisting of 99 objects, the child partitions each collection into sets of tens with a set of ones left over. The child then counts the total collection.

b. After doing a above, the child can (where possible) group the set of ones into another group of ten. The child also (where possible) classifies ten sets of ten as one set of one hundred.

c. The child can represent his actions in b (above) by using expanded numerals, i.e.,

$$\begin{array}{rl} 37 = & 3 \text{ tens} + 7 \text{ ones} \\ \underline{+\ 58} = & \underline{5 \text{ tens} + 8 \text{ ones}} \\ & 8 \text{ tens} + 15 \text{ ones} = \\ & 8 \text{ tens} + 1 \text{ tens} + 5 \text{ ones} = \\ & 9 \text{ tens} + 5 \text{ ones} = 95 \end{array}$$

d. The child can represent the actions in c (above) by carrying.

e. The child can interpret (explain) the symbols such as:

(1)

$$\begin{array}{r} 37 \\ \underline{+\ 58} \\ 95 \end{array}$$

by using expanded numerals as with objects.

f. The child becomes skilled in performing two-digit addition problems.

MATERIALS: One set of base ten multibase blocks, or an appropriate substitute that can be easily bundled into groups of tens and ones.

SETTING: These activities can best be accomplished with a small group (six to eight) of children seated around a table.

PREREQUISITES: Mastery of Grade 1 and 2 Numeration objectives.

MAP PRINCIPLES: The children will be actively engaged in manipulating and interpreting physical representations of multi-digit addition problems. Mismatch will occur as these situations are translated into written symbols.

ACTIVITIES:

Activity 1 (Exploratory Level)

Present the children with a set of tens and ones, where there are ten or more ones in the set. For example, three tens and twelve ones. Ask the children if it is possible to make another set of ten from the "12" ones. They should count out ten ones and replace it with one set of ten. When they have completed this, have them give the number of tens and ones they now have. In this case, it would be four tens and two ones. Now the children should be asked what number this represents. When they respond "forty-two," it should be noted that "forty-two" is the same as three tens and twelve ones. Continue this activity until the children become competent in making the appropriate exchanges of "ones" for "tens."

Activity 2 (Abstraction and Representation Level)

This activity should be considered an extension of Activity 1. Essentially, you go through the same process as in Activity 1; however, now you should record the procedure on the chalk board. The activity might go as follows:

Write the expression 5 tens + 15 ones on the board. Have the children represent this with the multibase blocks at the table. Have the children make the appropriate exchange of tens for ones and then ask them to explain what they did and what the result was. As the children explain the process, you should record it on the board as follows:

5 tens + 15 ones	(This is what we started with.)
5 tens + 1 ten + 5 ones	(You made 15 ones into 1 ten and 5 ones.)
6 tens + 5 ones 65	(Now we have 6 tens and 5 ones, the same as "sixty-five.")

This should continue until the child can make the correct exchange and write the process he has used.

Activity 3 (Exploratory Level)

Give the children two sets of multibase blocks, each representing a number less than 99. The children should count the number of tens and ones in each set and tell what number is represented by each set. Now have them combine the two sets and recount the number of tens and ones. Where necessary, the children should group all sets of ten ones into another group of ten and ten sets of ten as one set of one hundred.

Activity 4 (Abstraction and Representation Level)

Begin this activity as you did Activity 3 above. Ask the children what numbers are represented by the two different sets of tens and ones on the table. When they respond, record the numbers on the board.

46

38

Now ask how many tens and ones there are in each number. As the children respond, record this as follows:

46 = 4 tens + 6 ones

38 = 3 tens + 8 ones

Have a child combine the two sets of tens and ones and give the results. As the results are given, record this on the board.

46 = 4 tens + 6 ones

+ 38 = 3 tens + 8 ones

7 tens + 14 ones

Ask the children if there are enough "ones" to make another set of tens. As the answer is given, write the results on the board.

46 = 4 tens + 6 ones

+ 38 = 3 tens + 8 ones

7 tens + 14 ones =

7 tens + 1 ten + 4 ones

Now combine the 7 tens and 1 ten so the answer is complete, i.e., 8 tens + 4 ones or 84.

Activity 5 (Abstraction and Representation Level)

Present the children with a two-digit addition problem (written on the board).

67

+ 48

Ask them to solve the problem with the multibase blocks. As the children solve the problem, record each step of their solution as you did in Activity 4.

Activity 6 (Intepretation and Formalization Level)

This activity is similar in nature to each of the above, with one exception. In place of the expanded notation, used to represent tens and ones, we will use a "place value" short.

At this stage, the child may still be allowed to make use of concrete materials if necessary. As the children go through the addition process, have them explain what they are doing. In addition to this, you should record the stripe on the board similar to the following example:

Step 1

tens	ones
4	9
+4	7
8	16

Step 2

tens	ones
1	
4	9
+4	7
9	6

TEACHER NOTES: For all the activities discussed above, you may substitute other concrete aids for the multibase blocks. Sucker sticks, straws and rubber bands, are good substitutes. In Activity 2 you may wish to use an alternate activity. The process is the same as used in Activity 2, except the children can work on their own with little supervision. Have a stack of cards with sets of tens and ones printed on them; like the following:

Front	4 tens + 12 ones	52	Back

Have each child draw a card, perform the necessary regrouping to discover the number represented. The child with the largest number "has" the card. After all the cards in the deck are gone, a winner is the child with the most cards.

In each of the activities described above, care should be taken to present situations where it is not necessary to regrouping, i.e., 4 tens + 7 ones. This will help keep the children from automatically regrouping in every situation.

Activities 1-6 are appropriate for children at the stage of concrete operations. The preoperational child should be restricted to Activities 1 and 2. The transitional children can profit from Activites 1-5.

Multiplication

OBJECTIVE: The child learns the operational of multiplication of whole numbers.

a. The child can use the distributive, associative, and commutative laws for multiplication of whole numbers.

b. The child can use the inverse relationship between multiplication and division.

c. The child can use the multiplication facts for two numbers, a and b where $a \times b \leq 45$.

d. The child can count by two's, three's, four's, five's, six's, seven's, eight's, and nine's.

e. The child can use repeated addition to rename products.

f. The child can interpret a products, such as a x b, as a sets with b per set.

MATERIALS: Overhead projectors and a set of counting blocks.

SETTING: If necessary, this activity can be used with the entire class; however, it is more effective with a group of six to eight children.

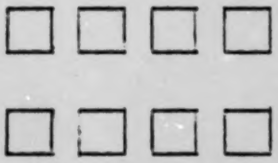
PREREQUISITES: Mastery of the Grade 2 multiplication objectives.

MAP PRINCIPLES: Mismatch will occur relative to the strategy employed by the students in solving the given problems. Self-regulation will be evident as the students vary in the time needed for counting, to a more "thought" oriented strategy.

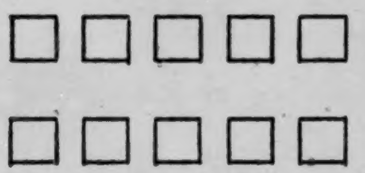
ACTIVITIES:

Activity 1 (Exploratory Level)

Seat the children so they all have easy access to the overhead projector. This will allow the activity to move along with less confusion. Begin by placing an array of blocks on the overhead projector. Let's use the following example.



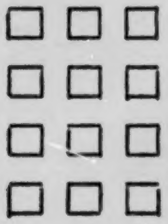
Have the children tell what multiplication problem can be solved by this model. Once they have stated the problem, they should also find the answer. It is possible to arrive at two multiplication problems from this illustration, 2 x 4 = □ and 4 x 2 = □. At this point, it is preferred that they interpret the array "row by column" or 2 x 4 = □. When the children give an answer, have them explain how they got it. This explanation may include: counting by "ones," counting by two's, or adding "4 + 4." In this development we want to encourage the latter strategy. Now present the children with a second example where you are multiplying by 2.



Elicite the same responses as you did in the first example and then continue. Follow this approach until the children can see, "if you want to multiply by 2, you can just add the two numbers, i.e., 2 x b = b + b.

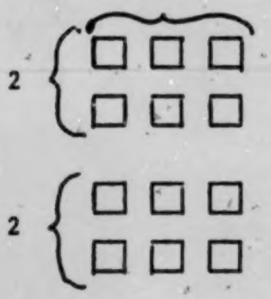
Activity 2 (Abstraction and Representation Level)

This activity will concentrate on multiples of four and does require that the child has performed well in Activity 1. Place the following array on the overhead projector.



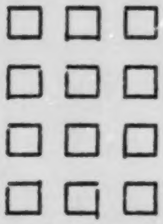
Ask the children what multiplication problem we can solve with this set of blocks. By this time the response will most likely be 4 x 3 = □. Follow up this answer by saying, "That's right. There are four sets of three or there are "four three's." Have the children find the answer to the problem 4 x 3 = □ and then discuss how they obtained their answer. After hearing their solutions, suggest the following strategy:

a. Rearrange the array on the overhead so it looks like this:



and then ask the children if there are still four sets of three. "That's right. Two sets of three and two sets of three is the same as four sets of three."

b. Now see if any of the children can "think" of an "easy" way of finding the answer to 4 x 3 = ☐. If no one suggests a way, explain as follows:



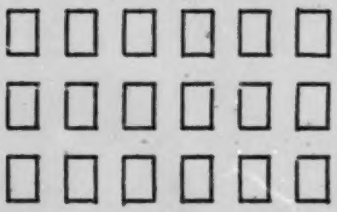
2 sets of 3 are 6 2 x 3 = 6

2 sets of 3 are 6 2 x 3 = 6

Continue this activity with the remaining multiplication problems of the form 4 x b. Make sure you concentrate on the strategy involved and not just getting an answer.

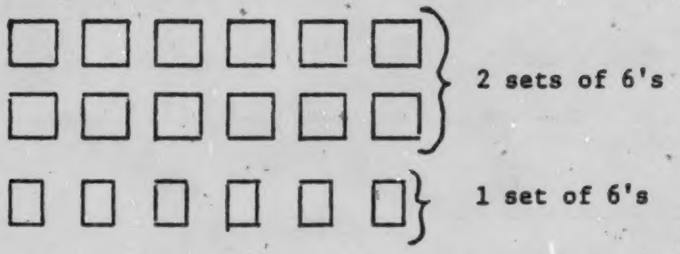
Activity 3 (Exploratory level)

This activity is very similar to Activity 2 in that we are trying to get the children to rely on prior knowledge. That is, rather than merely trying to rotely memorize the product of two numbers or having to always rely on repeated addition, the child should be taught a strategy to assist him in multiplication. For the purpose of this activity, products of the form 3 x b = ☐ should be used. Use the overhead projector to illustrate the following array of blocks.



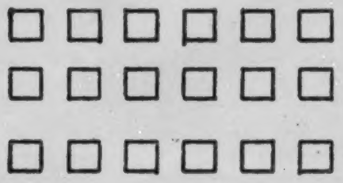
Again, the children should tell you what multiplication problem is illustrated and find the solution by any means they can. Once it has been stated that 3 x 6 = 18, continue as follows:

a) Rearrange the three sets of six so they appear as illustrated below:



Ask the children if there are still 3 sets of 6's. Then follow by saying that 2 sets of 6's and 1 set of 6's does make 3 sets of 6's.

b. Using the same illustration as in (a), ask if they could help in finding what 3 x 6 would be. If there is no suggestion, remind them of Activity 2 and try again. If there is still no answer, explain with the aid of the array of blocks.



2 sets of 6 are 12 $2 \times 6 = 12$

1 set of 6 is 6 $1 \times 6 = 6$

3 sets of 6 are 18 $3 \times 6 = 18$

c. Using a similar approach, illustrate and discuss the remaining products of the form $3 \times b = \square$.

Activity 4 (Abstraction and Representation)

This activity uses illustrations and questioning techniques similar to those used in Activity 1. The only difference should be that the illustrations would be accompanied by symbolic representation. For example, if the child is shown,

0 0 0

0 0 0

and responds $2 \times 3 = 6$, this should be written on the board and discussed with the children. You want the child to be able to interpret any array as a multiplication problem and given any multiplication problem he can illustrate and solve it with the appropriate array.

Activity 5 (Abstraction and Representation)

Present the child with the following types of problems,

0 0 0 0 0

0 0 0 0 0

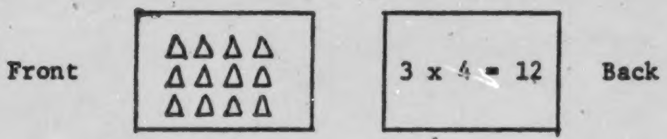
2 sets of 5 are _____

We write 2 x 5 = □.

You should be sure to accompany each problem with the appropriate pictorial representation. Before allowing the child to go into it alone, work several similar examples on the board.

Activity 6 (Interpretation and Formalization Level)

Make a set of 5 x 7 cards similar to the following:



Let a small group of children work around a table. A child draws a card and writes the equation represented by the array. Each child, in turn, selects a card until the deck is completed. Scores may be kept by each child and a form of competition can be attached to the activity if desired.

Activity 7 (Interpretation and Formalization Level)

Make a set of cards for facts like 1 x 4, 3 x 4, 4 x 4, 5 x 4, 6 x 4, 7 x 4, 8 x 4, 9 x 4. You may make as many sets as you wish, but the second number must be constant for each set. For example, you could have a set for a x 5, or a x 6, etc.

The front of the card should have an array illustrating a multiplication sentence and the back should have the sentence written in symbols form as in Activity 6. A child draws two cards from the deck and places it in front of him as illustrated below.

xxxx
xxxx
xxxx

xxxx
xxxx

He is then asked to write the one fact that would be represented by using both cards together. In our illustration, it would be 5 x 4 = 20. This could be taken one step further by presenting the back of the card and asking the same question.

3 x 4 = 12 5 x 4 = 20

TEACHER NOTES: This set of activities is by no means a complete multiplication development. You are expected to add many examples and extend the activities when you feel it is necessary. This activity set is directing itself to two basic concepts or ideas. First, the children should be taught that they can "fall back" on prior knowledge to obtain an answer and, second, the activities are being directed toward the use of the distributive property in multiplication. REMEMBER, the concern is not for the vocabulary, but for acquiring the use of the concept. In using these activities, it is suggested that you teach the multiplication facts in this order: one's, two's, four's, three's, and then five's. All of the activities are appropriate for the concrete operational child and all but the last two can be valuable for the transitional child. The preoperational child should be restricted to Activities 1 and 2.

Geometry

OBJECTIVE: The child learns the meaning of the parts of a polygon: vertices, sides, diagonal, inside, outside (or interior, exterior), and boundary.

a. Given an assortment of polygons, the child can count their vertices, sides, diagonal (often drawing them in), and locate points in the interior, exterior and on the boundary.

b. The child can use the terms vertices (vertex), side, diagonal, interior, exterior and boundary in working with polygons.

MATERIALS: Geoboards, rubber bands.

SETTING: These activities may be introduced using the entire class; however, groups of six to ten children will allow for better teacher-student interactions.

PREREQUISITES: The child should have acquired the concepts and skills described in Geometry - Grades K-2.

MAP PRINCIPLES: Self-regulation will occur in the complexity of the figure the child selects to work with. Content mismatch will occur as the child is presented with tasks involving more challenging figures.

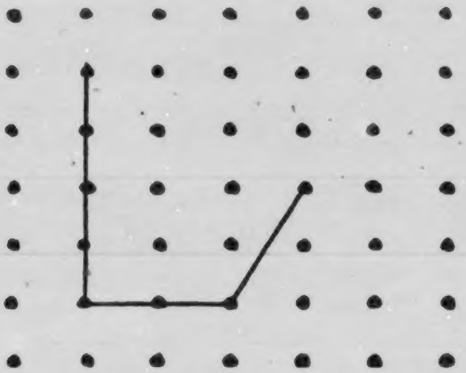
ACTIVITIES:

Activity 1 (Exploratory Level)

Provide each child with a geoboard and three rubber bands. Once the children have the geoboard, allow them a reasonable amount of free time to create any kind of picture they wish by encircling pegs with rubber bands. As this free play continues, it will lend itself to discussion of their pictures. Let the children describe the picture as they see fit; there is no reason to force mathematical description of, these endeavors. This exercise has no definite time schedule but is terminated as the teacher sees fit. Remember, we only want to familiarize the pupil with the apparatus and to provide for free discussion of what he sees and produces.

Activity 2 (Exploratory Level)

Each child should have his own geoboard for this activity. Ask the children to construct a triangle on their geoboards with a rubber band. Then each child, in turn, should be asked to place his finger on a peg (point) that is on the inside of the triangle. Ask the children how they know the point is inside the triangle. What they should see is that there is no way to "get out" without crossing the rubber band. Make a figure similar to the one below and ask if it has an "inside."



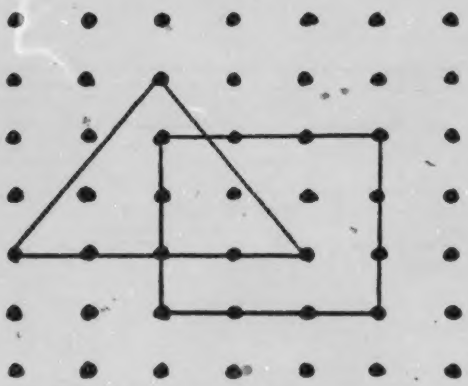
After allowing ample time for answers, discuss the responses. We want the child to see that this figure does not have an inside, because we can go from any point to another without crossing the rubber band. Continue this activity and discussion with as many different figures as needed to acquire the concept of "inside" a figure.

Activity 3 (Exploratory Level)

Use an approach similar to that employed in Activity 2, to teach the notion of outside and in a figure. The primary concern is that the child get the idea that the figure must be closed to have an inside and an outside. Also, the rubber band (line segments) represents the figure.

Activity 4 (Abstraction and Representation Level)

Construct a square and a triangle on your geoboard (see figure below).



Have the students (in turn) perform the following task. Make sure you discuss all responses (right or wrong) with the children.

Point to a peg . . .

a. inside the triangle

b. inside the square

c. outside the triangle

d. outside the square

e. inside the triangle and inside the square

f. inside the triangle and outside the square

g. outside the triangle and inside the square

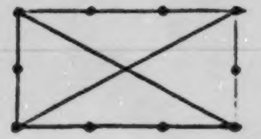
h. outside the triangle and outside the square

Activity 6 (Exploratory Level)

Have each child in the group make a simple closed figure on his geoboard. Discuss the figures and make sure they are "simple closed curves." (They are closed and they don't cross.) Have the students tell how many sides their figure has, if it is a special figure; triangle, square, rectangle, etc., have them give its name. Now have them count the corners on their figure. After they have given the correct number, tell the students that one of these corners is a vertex. Illustrate different shapes on the geoboard and have the students give the number of vertices for the figure. Continue the activity by selecting shapes in the room and finding the number of vertices it has.

Activity 7 (Abstraction and Representation Level)

Form a polygon (square, pentagon, etc.) on your geoboard. Have the students tell how many sides the figure has. Ask if anyone can use a rubber band to "connect" two vertices (without crossing one of the sides). In each case have them tell how many such lines there are in each figure.



These lines are called diagonals. Produce a number of examples and give each child an opportunity to place the rubber bands on the diagonals.

TEACHER NOTES: In this set of activities, the preoperational child should be restricted to Activities 1-3. The transitional and concrete operational child should be able to benefit from all of the activities.

Chapter VI

A Learning-Instructional Model for Mathematical Concepts

Up to this point, we have discussed content mismatch, mismatch in the level of abstractness of a problem, and mismatch in cognitive stage. Elaboration of these concepts is necessary in order that they can be meaningfully applied by the teacher in generating appropriate activities for children. To elaborate the concepts, a learning-instructional model is presented. The relationship of the model to teaching mathematics is analogous to the relationship of a blueprint to building a house. The principles that architects of blueprints use generally are taken from mathematics and from the sciences. The architect utilizes these principles in juxtaposition or in syntheses to formulate a sound plan. The builder, then, uses the blueprint to guide him in the construction of the house. But the blueprint in no way guarantees the quality of the builder's work nor the accuracy of interpretation of the blueprint. In a similar way, psychological principles are used in the construction of the model presented for teaching mathematical concepts. The model, however, in no way guarantees the quality of the learning of the children, for that is largely influenced by the quality of the work of the teacher and how well she can interpret the model.

A two-pronged approach is taken in the presentation of the model. First, factors contributing to cognitive development are elaborated and, second, learning-instructional stages are presented and their relationship to cognitive development made clear.

Factors Contributing to Development

Maturation

At least four factors have been identified which contribute to the development of cognitive growth of children -- and specifically, to the development of mathematics in the child. The first is maturation. In support of the proposition that maturation is involved in cognitive growth, it is a fact that transitive reasoning* has seldom been observed in children four years of age or younger. While that statement cannot be taken as proof that maturation is involved in development, it certainly indicates that maturation does play a prominent role. Because if it played no role in development, then subjecting a child to learning experiences would be sufficient for him to gain an understanding of the concept or principle involved. Evidence does exist, however, that great difficulty in learning transitivity of "same length as" exists for children in the stage of preoperational representation even when apparently appropriate learning experiences have been encountered. Thus, it seems likely that the changes occurring as the child grows older make possible learning which was not previously possible.

Experience

Experience by itself does not explain the conceptual growth of children but experience does play an important role in their conceptual growth. Too much variation has been observed in the age of attainment of the stage of concrete operations to discount the role of experience. But experience alone does not explain the growth of mathematical concepts. Otherwise, as already noted, all one would have to do to "teach" any child transitivity would be to give him sufficient experience -- and he would learn it! But, unfortunately, it is not that simple.

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*Transitive reasoning can be eaisly explained in the context of length of sticks. From knowing that stick a is the same length as stick b and stick b is the same length as stick c, one can infer that stick a and stick c are the same length.

Physical Experience and Mathematical Experience

Experience must be broken down into two components. One is physical experience and the other is mathematical experience. In order to make a distinction between these two types of experience, imagine that a child matches the objects of Set A one-to-one with the objects of Set B through overt actions -- he places one object from Set A with one object from Set B, etc., until all the objects of one or both of the sets are exhausted. Then he takes the objects of Set B and likewise matches them with the objects of another Set C. Now, does this matching constitute a physical experience or a mathematical experience? The answer is that it could be either one, depending on the child. One cannot differentiate between the two types of experiences through observation of the overt acts of matching in which the child engages. The crucial determiner of the type of experience is whether the Sets A and C are related by the child by virtue of the comaprisons of A and B and C. If the child is not able, through reasoning, to determine the relation between A and C, then the experience gained through overt matching of the objects of A and B and B and C was mainly physical in nature. The relation between the Sets A and B, in this case, was a function of the physical arrangement of the objects and would not exist for the child in the absence of the physical pairing of the objects. The relation would be external to the child and would be destroyed upon rearranging the objects of the sets. While the two sets of objects were in a state of physical comparison, the child could definitely obtain knowledge about the objects -- either they match or they don't -- but for the knowledge to be mathematical in nature, the relation must be conserved by the child when the objects are moved to new states, and the child must be able to engage in reasoning involving the properties (for example, transitivity) of the relations.

This distinction between a physical experience and mathematical experience is important in understanding the growth of mathematical concepts. What maturation contributes to this growth is the internal thought mechanisms which allow the child to go beyond physical experience. But one must hurriedly say that development must be supported by experience -- a child does not mature in a vacuum. The boundary between maturation and experience is not known -- that is, their relative contributions to the cognitive growth of children is not known. However, as has been emphasized, both are important to conceptual development.

More must be said concerning the distinction between physical experience and mathematical experience. In the case of knowledge derivable from physical experience alone, if a child is wrong it is easy to show that child he is wrong. Whereas in knowledge derivable from mathematical experience, if a child is wrong, it is generally quite difficult to show him that he is wrong; and indeed verbal transmission of the correct answer to the child is most often insufficient to show him he is wrong. For example, when overtly comparing two sticks, if a child fails to align two endpoints correctly, it is quite easy to correct his mistake. If, however, he fails to display transitive reasoning in a task, it is very difficult to demonstrate transitivity to him in one or two examples. In the case of transitivity of "as many as" given above, the relation between A and C has to be inferred by the child because the elements of A and C were not directly matched. There was no physical experience on which the child could rely. On the other hand, when physically comparing two sticks, the child can be shown through a physical action if he is wrong in his alignment of an endpoint. Another example is where a child has eight wooden beads, three white and five brown. If he is asked whether he has more brown beads or more beads, and if the child errs, again it is quite difficult to show him he is wrong. Knowledge acquired through physical experience alone is well worth knowing, however, and often is the source of observations leading to more organized knowledge.

Knowledge derivable from physical experience alone is called physical knowledge. Physical knowledge is characterized by knowledge about the properties of objects. Physical experience is generally thought of as experience through direct contact with objects through one of the five senses. For example, one may touch something and it is hard, cold, hot, soft, supple, etc. Or, one may see something -- an object is red, a diamond cutting glass, the shape of a banana, etc. Knowledge gained through these observations is concerned with the properties of the objects. An observer, however, may bring something to the observation which allows him to go beyond physical knowledge and gain logical or mathematical knowledge about the objects not possible by another observer. An example is that while redness is a property of an object which is, in fact, red, a property which exists independently of the observer, light can also be described as wave motion, so that an observer who knows this may experience the redness at the level of mathematical experience. Another example is where an observer sees an iron boat floating and another piece of iron sink. These two observations can be summarized by saying "sometimes iron floats and sometimes it sinks."

This knowledge is pure physical knowledge if the implicit contradiction is not removed. Its removal demands a mental construction beyond the physical knowledge gained through direct observation. Knowledge gained through observation alone is at best fragmentary if not connected by principles.

Even though an experience which is the result of direct contact with an object through the senses may go beyond physical experience (depending on the observer), the source of mathematical experience is generally thought of as being overt actions. This does not mean that just because a child is involved in overt actions, he will be having a mathematical experience! It has been already noted that a child may overtly match the objects of two equivalent collections and the knowledge gained from the overt matchings be nothing more than physical knowledge. This is especially true for children at the stage of preoperational representation and those capable of only intuitive one-to-one correspondence. Those children capable of one-to-one correspondence would, by definition, gain mathematical knowledge from the overt actions. A critical difference is that the mathematical knowledge gained demands that a pair of physical objects not be defined by the closeness of the objects. Two objects may make a pair even though they are quite far apart! An example often cited of mathematical experience is where a child realizes that it makes no difference how you count a collection of objects -- you get the same number. This knowledge is gained through counting the collection in at least two ways.

Linguistic Transmission

Another factor contributing to the growth of mathematical concepts is linguistic transmission of information (information transmitted through language, oral or written). This factor includes the verbal interchange of the child with other people. As such, it is to be considered as a part of the experience of the child. However, experience goes beyond linguistic transmission, so what is said about the latter is not necessarily true of the former.

Certainly, a child can receive valuable information via language. One must not think however, that information contained in a verbal communication will necessarily increase a child's understanding of a mathematical concept. To verify this statement, a transitivity problem was presented to 40 first graders, 40 second graders, and 40 third graders who were in the top two-thirds of their class according to teachers' judgments. Each child was presented with a transitivity problem involving length relations. If they could not solve the problem, they were told the correct relations which held between the two sticks. For example, if a child took a stick B and compared it with C and then with A and found that A and B were the same length and B and C were the same length, then could not infer the relation between A and C, they were told that A and C were the same length. After being told, the child was asked to explain why A was as long as C. Of 24 children who could not infer the correct relation between A and C, only five could explain why A and C were of the same length after being told. Even though the children were not told the reason A and C were of the same length, they had just gone through the two comparisons and said that A and B and B and C were of the same length. Being told that A and C were of the same length was not sufficient for 19 of the children to mentally go back over their actions and gain information from reflecting on them. The actions became significant for only five of the children.

While the above example is quite specific, it does illustrate the point being made. It is legitimate to say that for a child in the stage of pre-operational representation (or even in the transitional stage), attempting to teach him mathematics by only verbal or symbolic means is in danger of leading to disasterous results. But, because words and symbols are a part of mathematics teaching, their role must be further clarified. Experience has taught us that there should be a continual interplay between the spoken words which symbolize a mathematical concept and set of actions a child performs while constructing something that makes the concept tangible. In short, there is good reason to develop mathematical vocabulary during the course of activities used to develop the concept. The particular blend has to be detemined by the specific activity and child engaging in the activity. But it is a long way from promoting vocabulary development to recommend that the teaching of mathematics to young children be based on symbols of mathematics or verbalization.

Equilibration

The last factor important in the growth of mathematical concepts is a principle called equilibration. Of the fours factors which contribute to the growth of mathematical concepts, this factor is the most fundamental but the most difficult to employ in practice. There is not much one can do to vary maturation, short of being sure the child is physically healthy. So, while one must acknowledge and understand maturation, essentially there is little a teacher of school mathematics can do to control it. Experience and linguistic transmission are under control of the mathematics teacher, but only insofar as the mathematical experiences of the child are concerned. Here it is crucial to distinguish the different levels of experience and appreciates the role of language in planning the mathematical environment for the child.

But little has been said yet about knowledge acquisition -- that is, is there anything which would help to understand how a child acquires mathematical knowledge? In the material presented on relations, it was seen that either children have little or no knowledge of relations, or they are able to engage in reasoning involving the properties of the relations, or they are oscillators -- at times, in restricted situations, they appear as if they are able to reason involving relations, but that reasoning is limited and can be extinguished quite easily. The difference between physical experience and mathematical experience helps to clarify the role of manipulative activities in the classroom. Some children may engage in a manipulative activit, but yet be involved only at the level of physical experience, whereas another child may be involved at the level of mathematical experience. But the question of how to maximize the possibility of a child's engaging in the activity at the level of mathematical experience remains yet unanswered. Stated another way, are there any clues which one could use in taking a child from a physical experience to a mathematical experience? One of these clues is equilibration of self-regulation.

Assimilation and Accommodation. Equilibration consists of two parts. One of these parts is called assimilation and the other is called accommodation. Assimilation is thought of as an integration of any sort of reality into an existing mental structure. Accommodation is thought of as the complimentary process of assimilation. When one integrates new information, an alteration may take place in the existing mental structure. This alteration may be the creation of a new mental structure or a modification of already existing structures. Accommodation is not only thought of in terms of alteration of structure, but also as application of available mental structures to environmental input.

It must be stressed that self-regulation or equilibration occurs as a result of mental or physical activity on the part of the child. Acquisition of knowledge is not possible in the absence of activity. We speak of assimilatory activity and accommodatory activity -- there is no way that reality can be integrated into an existing cognitive structure in the absence of mental activity. Examples of assimilatory activity are play in its more practical sense (e.g., freeplay with toys) or in its most symbolic or structured sense (e.g., rule-bound play with games which represent reality -- monopoly) or learning in its broadest sense. Examples of accommodatory activity are memory and imitation.

Learning. Learning is usually thought of as being provoked by a situation external to a person. This does not mean that mathematical learning can be accounted for by just a stimulus which is to provoke a response. An example of such a situation is "2 x 3" as a stimulus and "6" as a response. The association of 6 with 2 x 3 is important and can be conditioned (or memorized) using appropriate teaching strategies. If this were all mathematical learning consisted of, however, it would be indeed a sterile subject. The stress in mathematical learning would be on "answer getting" and little attention would be paid to the interconnectedness of the subject. It is critical, even in the very beginning, to stress the interconnectedness of mathematics in the learning of it. Teaching which assumes only a stimulus-response view of learning runs a great risk of promoting only physical knowledge and not mathematical (or operational) knowledge. However, because learning is thought of as being provoked by situations external to the learner, it is necessary to further analyze levels of learning particular mathematical concepts or structures.

Levels of Mathematical Concepts. Number, relations, and classes take a relatively long period to develop as operational concepts in the child. Their development may be even as late as nine years of age in some children.

The stages of development of these concepts have been called preoperational, transitional, and concrete operational stages. One basic assumption in this document is that most mathematical concepts go through quite similar "stages" (called levels) for the person who is learning the concept -- that is, there is essentially no concept, then rudiments of the concept exist, and then the concept becomes "operational." However, for many mathematical concepts these levels should not be confused with the stages in development, even though they are quite similar. One essential difference is that stages of natural development are the result of a child's interaction with his total environment and occur in every sane person. But for most mathematical concepts to occur, special learning environments must be created. The creation of the learning environments does not insure that a concept will be learned, but there are learning-instructional stages which are most natural for the three rather gross levels of mathematical concepts identified.

Learning-Instructional Stages for Mathematical Concepts

Exploratory Stage

The first learning-instructional stage is called Exploratory. Children's play is considered as a critical part of this learning-instructional stage. Play is viewed as an assimilatory activity and is an essential part of early mathematics learning. Whenever assimilation occurs, its counterpart, accommodation, also occurs if self-regulation is to happen. Self-regulation, or equilibration, is viewed as one essential factor of development, and is now extended to learning mathematical concepts in general.

Self-regulation, or equilibration, is a useful theoretical construct to help guide learning activities in mathematical instruction. By itself, however, it provides few hints in specific situations. More is needed in order to employ the principle in practice. Play activities, as a teaching technique, should be thought of as corresponding mainly to the first level of mathematical concepts identified -- essentially no concept -- and to the second level -- rudiments of the concept. At these two levels of mathematical concepts, a great deal of constructive thinking needs to be done by the child. It is a period of concept formation, not analysis. More analytical thinking must come after something exists to be analyzed.

Play activities can vary along two quite important dimensions. The first is the type of material and the second is the external direction the child is given. The materials can vary from structured to unstructured and a play activity can vary from highly directed to undirected. In the latter case, it is important to realize that the child usually structures his own play activities.

Multiple Embodiment Principle. In order to illustrate the principles of self-regulation and play activities using particular concepts, imagine that a teacher decides that one-to-one correspondence is to be worked on. In this case, no concept of one-to-one correspondence corresponds exactly to the preoperational stage of development. Many problem situations have already been given which can be used by the teacher to approximate which of her children are without a concept of one-to-one correspondence. After such an approximation is made, the teacher should allow the children who do not display any concept of one-to-one correspondence to engage in undirected play activities, using physical objects which will later be used in directed play activities. For example, the teacher may have assortments of beads, bird cutouts, blocks, discs, animal cutouts, toy animals, toy cowboys, toy soldiers, toy guns, dolls, dresses, toy dishes, toy utensils, etc.

If the children are allowed free play time with these materials, they generally will engage in practical or symbolic play. For example, almost any child will attempt to place cowboys and Indians on horses, dress dolls, stack dishes, stack blocks, string beads, categorize animals or bird cutouts, align soliders in rows and give each guns, etc. This type of practical play may be extended by the children into more symbolic play, such as waging wars, keeping house, setting table, etc. The length of time and the number of free-play activities which should be encouraged by the teacher is open to debate. But there is a principle involved which the teacher can use. This principle is called the multiple-embodiment principle. It states simply that in play activities, the child should use as many different material sets as possible where each material set is conducive to construction (by the child) of a concept.

Let's take a particular free-play activity where the preoperational children involved place cowboys and Indians on horses. Through this assimilatory activity, the children can gain the physical knowledge that indeed the cowboys and Indians fit on the horses! The teacher can now employ an artful suggestion which may create a disequilibrium (or mismatch) for the children. For example, she may suggest that the children find if there are enough cowboys and Indians so each horse would have a rider. In order to find out, a child has to accommodate his practical or symbolic activity to engage in a goal directed activity. If the children do so, the teacher can employ the multiple embodiment principle and give them new materials with a similar goal (are there enough dresses so one could be put on each doll?).

If the children do not initiate their own goal directed activity upon suggestion, she may directly show them how to find out if there are enough cowboys and Indians so each horse has a rider. The children, then, through imitation (or accommodatory activities) can answer the question. Here, again, the principle of multiple embodiment is important because the teacher certainly wants the child to initiate such activity in any appropriate situation and may wish to have the children employ imitative behavior in more than one situation, if necessary. The teacher must be sensitive, however, to the type of knowledge the child is acquiring in these imitative activities. The knowledge acquired has a high probability of being physical knowledge with children in the stage of preoperational representation. While this should not alarm the teacher, if she is sensitive to it, she will not attempt to try to build higher-order concepts on one-to-one correspondence.

Mathematical Variability Principle. The principle of multiple embodiment is not the only principle the teacher can use in managing play activities of children. It has already been pointed out that while children engage in free play activities, the teacher, through artful intervention, can take the free play activity and change it into a directed activity for the children. If the teacher's suggestion fails to transform the free-play activity into a directed activity, she can try other suggestions or a direct demonstration, such as placing cowboys and Indians on horses, and having the children engage in imitative activity. Beyond these suggestions, the teacher can employ what is called the mathematical variability principle. In the multiple embodiment principle, the mathematical content is held constant and the materials varied under the constraint of being conducive to construction of the concept by the child. In the mathematical variability principle, the mathematical content is varied. In case of one-to-one correspondence, the teacher can vary the relation being considered to either a new relational category altogether (e.g., length relations; family relations) or vary the relation within the category of matching relations (more than, fewer than, as many as). Both of these types of variations can be used to create disequilibrium in the child, so that self-regulation is given an opportunity to operate.

The mathematical variability principle and the multiple embodiment principle both have been explained in terms of the play activities of children. These play activities are viewed as the first learning-instructional stage in a cycle of three learning stages for mathematical concepts. The two other learning-instructional stages are: a stage of abstraction and representation, and a stage of formalization and interpretation. Before characterizing these latter two stages, additional comments are made about the exploratory stage.

Within the exploratory stage, it has been pointed out that play can vary from free play to directed play, where the directed play is a natural extension of free play. Moreover, the teacher can employ the principles of multiple embodiment and mathematical variability in attempting to take a child from physical experience to mathematical experience. But remember that maturation also contributes to the development of certain mathematical concepts (classes, relations, and number, in particular). Consequently, the teacher should not expect dramatic short-term success in teaching those topics to preoperational children. She can expect more success with children in the transitional stage (which corresponds here to the second level of mathematical concepts). But again, the short-term success will undoubtedly be modest. With children in the stage of preoperational representation, it is advocated that the teacher hold the learning stage constant (i.e., use the exploratory stage!) and employ the mathematical variability principle, the multiple embodiment principle and utilize the free play-directed play distinction. It is felt that it is more humane to utilize a wide variety of mathematical concepts in multiple contexts than to attempt to take the children to the higher two learning stages with regard to particular concepts. This opinion is predicated on the assumption that the preoperational children engaging in play activities will encounter only physical experience in most cases. For the higher two learning-instructional stages, the children must be able to acquire mathematical knowledge -- which is another way of saying they must be able to engage in mathematical experience. At the exploratory stage, it is also advocated that the mathematical language specific to the mathematical concepts dealt with be developed.

Abstraction and Representation

The second learning-instructional stage identified, that of abstraction and representation, is based in part on the distinction between physical experience and mathematical experience. In the case of physical knowledge, abstraction can and does occur, but it is abstraction about properties of objects and does not lead to mathematical knowledge. Abstractions of this nature are called simple abstraction. Some examples are hardness, sharpness, etc. But another type of abstraction exists and is called reflective abstraction which is abstraction from the actions performed on objects or representations of objects. Knowledge gained through reflective abstraction is called mathematical knowledge. An example is where a child counts a string of beads from one end, then from the other and realizes that the number of beads is independent of the manner of counting them -- i.e., which is selected first, which second, which third, etc. The beads are there, but the knowledge gained had to do with the actions of the child and his capability of gaining knowledge through going back over his actions and realizing their significance or, in other words, reflecting on them. Another example is where a child pairs elements from two sets until one is exhausted before the other, and then pairs them again a different way, whereupon he realizes that it makes no difference how the pairing is done, the one set will always contain more elements than the other. Still another example is where a child compares stick A and stick B, then B and stick C and deduces that A is longer than C based on the two initial comparisons. Other examples can be given, but are unnecessary to illustrate the meaning of reflective abstraction. It should be clear that a child may be engaging in play but still engage in reflective abstraction. There is no contradiction here. It must be remembered that the child who does engage in reflective abstraction has the potential of going quite beyond play activities in the sense of the learning-instructional stage called exploratory. While it is entirely possible to continue in mathematical-like games for such children, these children can engage in much higher level games (insofar as mathematical concepts are concerned). But the teacher need not be restricted to game-lik activities in her teaching of these children.

The child can engage in reflective abstraction, but yet, not make a representation of newly gained knowledge. A representation can be a drawing, a diagram, a collection of symbols, etc. For example, if a child compares a green stick with a red stick and finds the green stick to be shorter than the red stick, G < R could be used as a presentation; $\underset{G}{|} < \underset{R}{\Big|}$ could be used as a representation; or $\underset{G}{|} \quad \underset{R}{\Big|}$ could be used as a representation, among others. So, when a child engages in reflective abstraction and representation, he is definitely operating at a higher level than he was expected to in the first learning-instructional stage. Rudiments of mathematical concepts being dealt with exist so that the second level of mathematical concepts is about all that is expected here.

Formalization and Interpretation

The learning-instructional stage of formalization and interpretation completes the cycle of learning mathematical concepts. In order to illustrate this learning-instructional stage, the mathematical concept, base ten numeration system, is selected and the three learning-instructional stages illustrated and differentiated.

We will assume that the child is at the concrete stage of operations in development so that operatory classification and relations* are at his disposal. It must be made clear that just because a child is at the stage of concrete operations, he will not necessarily know base ten numeration or even have made representations of his knowledge. What he is able to do with classes, relations, and number has in no way necessarily been formalized. This knowledge is largely unconscious for the child. However, we will also assume the child has just completed a learning cycle concerning the digits 0, 1, 2, 3, 4, 5, 6, 7, 8, and 9, can write them, order them, and do simple addition.

Any natural number can be written in expanded notation. For example, 326 in expanded notation is $3 \times 10^2 + 2 \times 10^1 + 6$. The 3, 2, and 6 are called coefficients, 10 is called the base, and the 2 and 1 in 10^2 and 10^1 are called exponents. Consequently, the coefficients, the base, and the 3xponents can all be allowed to vary in employing the mathematical variability principle. Usually, the base is held constant (base ten is used) and only the exponents and coefficients are allowed to vary. This philosophy is adhered to in all except the very beginnings of instruction in the exploratory stage.

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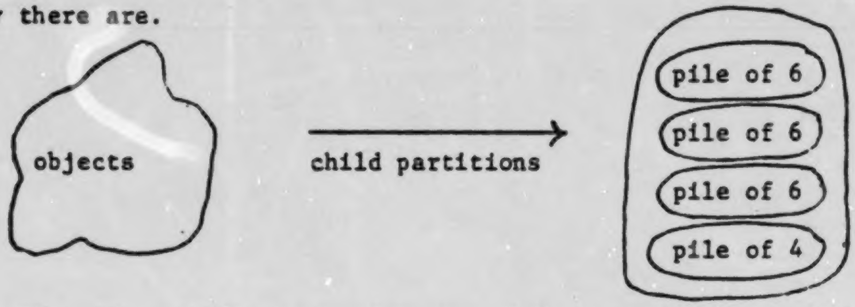
*See Chapter VI.

Imagine that a small group of children are given a collection of various assortments of materials, such as geometrical shapes, checkers, dried beans, etc. The children are allowed to engage in free play with the materials, building whatever they wish -- castles, hosues, roads, forts, etc. Because these children are at the stage of concrete operations, the teacher is able to intervene with artful suggestions to direct play activities. The first type of suggestion she should make is to have the children find how many piles with a certain number in each pile they can make. She can employ the mathematical variability principle to vary the number in each pile, or the total number of objects in each collection. The total collection should not contain more than, say, 40 to 50 objects, or the children will quickly tire of the task. The multiple embodiment principle can also be employed in at least the following ways. The type of objects can be varied, thus setting a new problem each time; or the type of collection formed by the children can be varied. For example, strings of beans with, say, five per string; stacks of blocks with, say, ten per stack; plates of dried beans with ten per plate; etc., can all be used. The essential thing being worked on here is that a collection of objects can be partitioned into subcollections with the same number in each subcollection and one other subcollection with fewer objects in it than in any other subcollection -- a collection of twenty-six objects can be partitioned into four subcollections with six per subcollection and two more.

In the first few partitions, the children will most probably participate at the level of physical experience. One of the first bits of mathematical knowledge the children should acquire is that there is the same number of objects in the total collection before partition as there is after the child has partitioned it. That is, the child should be able, through his actions, determine that a pile of objects can always be put back the way it was before the partition, that no objects were added or subtracted, so the number of objects before piling is the same as after piling, without actually knowing how many there are.



Specifically, if a child starts with some objects, makes three piles with six per pile and one pile of four, the child should know that the total number of objects in the original pile is the same as the number of objects in three piles of six and one pile of four, without knowing there are 22 objects. If a child realizes this, he just made a transition from the first learning stage (exploratory stage) to the second learning stage (abstraction and representation). Hopefully, by using the mathematical variability principle and varying the number of objects in each pile, the child will realize that no matter how many are in each pile, the total number in all the piles is the total number of objects.

It should be clear that the child is well on his way to constructing the concept of a numeration system. While he has a lot of information yet to acquire, the operational basis for further work has been laid. The basic goal in the second learning stage is to have the child construct a notational system and construct the place-value concept. Generally, the two-digit numbers are worked on at different age levels than are the three digit numbers, which in turn are worked on at different age levels than are the four digit and higher digit numbers (the generalization of place-value).

After the children first enter the learning-instructional stage of abstraction and representation, they are able to partition a collection into subcollections and know that the number of objects in the original collection is the same as that in all the subcollections. Capitalizing on this knowledge and the ability of the children to engage in rational counting, place-value concepts may be developed. Give the children a collection of objects, say 35, and have them count out a pile of ten and place them in a transparent bag; count out another pile of ten and place them in a transparent bag; and then note there are only five remaining. The children then should fill out the following open sentence: ________ tens and _________ ones. Various activities such as the above can be done until the children are ready, in the teacher's estimation, to take one further step and write, for example, the symbol "35" to represent 3 tens and 5 ones. Another activity at this second learning-instructional stage which is useful later on in subsequent learning is the tally chart. The tally chart may be thought of as a representation. For two digit numbers, it looks as follows. In the particular tally chart given, the two marks under "tens" means two sets of ten have been counted and the mark under "ones" means there is a single element remaining in the total collection. It should be clear that the tally chart is a representation and can be used to represent any number from 0 to 99, inclusive. Use of the tally chart should be coordinated with the development of the numerals. The most singular difficulty children have with the tally chart is that they

tens	ones
11	1

forget the marks in the tens place have a different meaning than do the marks in the ones place.

In the activities immediately above, it should be emphasized that the number names are not mentioned, only the symbols, such as "63" are mentioned, and they are to be interpreted as six tens and three ones. After children can represent any collection, such as a tens and b ones, where a and b are digits, as "ab," they are ready to learn the number names for two digit numbers and order the numbers from 0 to 100. We are here entering the next learning stage because the knowledge gained to this point is going to be systemized by the children. The basis for the learning which is to take place has been laid in counting out piles of ten. However, the main goal of the next learning stage is to, again, systemize the whole numbers from 0 to 100 using the number names. Other learning cycles will be built on this cycle -- such as cycles having to do with counting by two's, three's, four's, five's, six's, etc.; addition, multiplication of two digit numbers and their properties; and subtraction and division.

To initiate this new learning cycle, have the children count out a collection of objects as the basic learning mechanism. When a pile of ten has been found and collected together, then one more ten is counted out. The two tens are symbolized by "20" and the spoken number is given. The decades are developed in this manner and the symbols "<" and the phrase "less than" is introduced or reviewed, as the case may be. The children then should work on ordering the decades arriving eventually at $10 < 20$; $20 < 30$; $30 < 40$; $40 < 50$; $50 < 60$; $60 < 70$; $70 < 80$; $80 < 90$; and any other variation, such as $20 < 70$. The tally chart is a useful tool in developing the above activities. The children should not only be able to write "20 < 30," but also say "twenty is less than thirty," and be able to complete open sentences such as __ < 40 correctly (10, 20 or 30 are all good answers). The mathematical variability principle is widely used in all of the above activities in the second and third learning-instructional stages. The multiple embodiment principle is especially used in the second stage but to a lesser extent in the third stage. This is natural due to the character of the activities and the level at which they are conducted. In the third stage, however, the multiple embodiment principle is most used in interpretation, which has not yet been mentioned.

Formalization is taking place in the sense that a notational system is developed and organized by the child. The organization of the notational system is based on the abstraction and representation accomplished at stage two and on the new element of an order relation. The order relation is an essential part of the third learning-instructional stage for the concept of numeration. Without it, the third learning stage would have little meaning. The order relation, however, is based on one-to-one correspondence, so that preliminary learning cycles will have to have been completed with regard to one-to-one correspondence and number.

After the decades have been symbolized and ordered, and the child can count by tens, the decades can be completed based on the relation "one more than" just as the decades were ordered on the basis of "one more then than." Eventually, we want the child to be able to say the number names for all the numbers from 0 to 99 inclusive, be able to write the numerals, order any two of them, have the complete sequence ordered, and be able to count by ones from 1 to 99 inclusive. In the ordering tasks, the child should know that, for example, that any number in the "sixties" is greater than any number in the "forties," so that no confusion exists in ordering two numbers such as 47 and 63.

The above learning cycle with regard to numeration will undoubtedly be interrupted by other learning cycles. In fact, it is advocated that two or more learning cycles be operating so that boredom is decreased. As a teacher you know various strands of the mathematical curriculum exist. These strands can be used as guidelines for the creation of learning cycles just as can the sequence in standard textbooks. However, do not mistake textbook sequences to display the learning cycles you may want. They can be used, however, in employing the mathematical variability principle. The teachers' manuals often offer invaluable suggestions for learning activities, just as does this Strategy Manual.

Summary

For particular learner, it is assumed that mathematical concepts go through three levels -- essentially no concept, then rudiments of the concept, and then an operational concept. These three levels of concepts form the basis for identifying three learning-instructional stages for mathematical concepts -- the exploratory stage, the stage of abstraction and representation, and the stage of formalization and interpretation. These learning-instructional stages interact with the type of experiences and the cognitive stage of the child. Mathematical concepts which have been shown to be developmental in nature (number, relation, and classes) need to be considered different than concepts which have not been shown to be developmental in nature. For the case of the latter category of concepts (numeration, addition, subtraction, multiplication), if a child is preoperational, then the teacher should not force the child to go beyond the exploratory learning-instructional stage. The only type of experience such a child is capable of is physical experience. It must be stated explicitly that the type of experience of a child is not under the control of the teacher. She can give children the opportunity to engage in mathematical experiences, but there is no way she can force the child to engage in mathematical experiences. Moreover, the child has little or no conscious control over which type of experience he engages in. It is something which just happens, and, to a large extent, depends on the cognitive stage which the child is in. So, for preoperational children, the teacher should not expect the children to go beyond physical experience in the exploratory stage. She can aid the child (or give the child the opportunity) to engage in mathematical experience through employment of the multiple embodiment principle and the mathematical variability principle, but she cannot make the reflective abstraction for the child. Through the process of self-regulation, the child will eventually realize the significance of his actions and thus enter the next learning-instructional stage with regard to particular concepts. Through maturational processes and experience, preoperational children will move to the concrete operational stage and thus become much more likely to engage in reflective abstraction.

Concrete operational children have much more mathematical learning potential than do preoperational children. These children will begin, with regard to particular concepts, with play activities just as do the children in the two other stages. While they may begin with physical experience, they can do quite beyond physical experience to mathematical experience, and hence to the higher two learning-instructional stages. If a child is operating at the stage of abstraction and representation, with regard to a particular concept, there is no way that he will engage only in physical experience because of the very definition of that stage. The same can be said for the next higher stage.

All children, then, whether they are concrete operational, preoperational, or transitional, should be given the opportunity to learn the same mathematical concepts. The depth of concept learning will depend to a large extent on his stage of development -- so that a teacher should expect concrete operational children to complete learning cycles

lower stages of cognitive development should not be expected to complete the learning cycles, but should be given the opportunity do so.

One further comment is in order concerning physical experience and mathematical experience. One should not equate physical experience with physical objects, or physical actions on such objects, and mathematical experience with abstract thought. A child may engage in mathematical experience through manipulation of physical objects, physically or mentally. The presence or absence of the objects just does not determine whether or not a child engages in mathematical experience. In fact, presence of objects may facilitate a mathematical experience -- but the presence of objects is not logically necessary for a mathematical experience. Physical actions are not necessary for a child to engage in physical experience -- but some physical context is. So, whether a child engages in physical experience or mathematical experience depends ultimately on the child.

Tabular Representation
of a Learning-Instructional Model:
Types of Experience

L-I Stage / Cognitive Stage	Exploratory	Abstraction & Representation	Formalization & Interpretation
Concrete Operational	Physical and Mathematical	Physical and Mathematical	Physical and Mathematical
Transitional	Physical and Mathematical	Physical and	
Preoperational	Physical		

In the tabular representation of the learning-instructional model, concrete operational children are capable, at each learning-instructional stage, of engaging in physical or mathematical experience. It must be emphasized that a child's being in the stage of concrete operations does not guarantee he will be able to complete a learning-instructional cycle with each concept presented. However, it would be unlikely that a child who did complete a learning-instructional cycle would be a preoperational or transitional stage child, especially a preoperational child. In fact, it is not assumed that a child in the preoperational stage is capable of making a reflective abstraction. Children in the transitional stage should be expected to be capable of making reflective abstractions and representations, but not necessarily be capable of organizing their knowledge in the sense of the formalization and interpretation stage. Preoperational children should not be expected to be capable of engaging in mathematical experiences. It should be emphasized, however, that there may be exceptions to the patterns outlined in the tabular representation.

Mathematical concepts which are developmental in nature for the child (number, classes, and relations) essentially do not "fit" into the tabular representation given. For these concepts, the learning-instructional stages and the cognitive stages are essentially not distinguishable. Consequently, instruction on these concepts should not be expected to be successful in a short-term sense. The reason for this state of affairs is that maturation is not a factor in the learning of nondevelopmental concepts, given the child is in the concrete operational stage. However, for concepts of numbers, relation, and classes, if a child is concrete operational, he essentially is operational with the concepts, but may not have symbolized them.

Chapter VII

On the Child's Conception of Number

Teachers of mathematics often do not perceive the potential contribution of cognitive development theory to teaching mathematics. But if all a child learned about mathematics had to be taught through school instruction, education in mathematics would be forced to be much more efficient than it now is. However, it is a mistake to assume that children begin school with little or no mathematical knowledge and acquire such knowledge from school instruction alone. Various mathematical concepts have been shown to develop through interaction of children with their total environment and can be considered to be a part of the basic intelligence of children at certain stages of their intellectual growth. It is important for teachers to know the basic stages of intellectual growth of children and ways children conceive of mathematical concepts at different stages for the following reasons.

1. In many cases, what a child learns from a particular bit of instruction is influenced by the stage of intellectual growth of the child.
2. Often a child does not think about a particular topic in the same way as does an adult. An adult, nowing this, may actually inhibit the child's attempt to understand by imposing thinking patterns on the child in a highly symbolic form.
3. Because stages of intellectual growth are characterized by using mathematical-like concepts, an adult is able to gain insight into the thinking of the child in mathematical situations by understanding stages of intellectual growth.

The stage of preoperational representation begins around 18 months of age and lasts until around six to seven years of age. Obviously, dramatic changes take place in children during this time -- the most dramatic perhaps is language development. Only the upper end (four to six years of age) of the stage is of concern for mathematics teaching in preschool or early elementary school. The stage of concrete operations begins around six to seven years of age and lasts until about 11 to 12 years of age. It must be emphasized that the age break between the two stages varies from child to child. One cannot expect age to exactly determine stage, as some children do not reach the stage of concrete operations until after eight years of age, whereas some children may reach the stage as early as five years of age. The essential thing to realize is that every child goes through the stages in the same order but not necessarily at the same rate.

In the following material, the stages of intellectual growth are presented by selecting particular mathematical concepts and describing known ways that children deal with the concepts. This approach has three advantages. First, the teacher can contrast the way in which she thinks about the mathematical topics with the way in which children think about them. Second, the teacher can gain valuable techniques for presenting the topics to children. Third, the teacher can gain some additional mathematical knowledge.

RELATIONS

One-to-one correspondence

The notion of one-to-one correspondence is at the heart of number. Two collections of objects, say a jar of pennies and a jar of lemon drops, may be compared in order to find if the two jars contain an equal number of objects, of whether one has more than the other, through successively pairing a penny with a lemon drop until the objects of at least one of the two jars is exhausted. If the objects of both jars are exhausted simultaneously, then it can be concluded that there are the same number of pennies as lemon drops. If one jar is exhausted before the other (say the lemon drops are exhausted before the pennies), then there are more pennies than lemon drops *and* fewer lemon drops than pennies.

For another example, consider the objects of {Δ, √, *, v} and the objects of {a, b, c, d}. One pairing of the objects of these two sets is Δ↔a; √↔b; *↔c; and v↔d. Another example is a collection of dogs consisting of a Poodle, a Shepherd, a Collie, a Terrier, and a Huskie; and a collection of cats consisting of a Persian, a Siamese, a Manx, and a Calico. The pairing: Poodle↔Calico; Shepherd↔Manx; Collie↔Siamese; Terrier↔Persian is a one-to-one pairing which does not exhaust the collection of dogs but does exhaust the collection of cats. In this case, the collection of cats is in one-to-one correspondence with a subcollection of dogs (but not all the dogs) and the child can conclude there are more dogs than cats and fewer cats than dogs.

Children's conception of one-to-one correspondence

Children in the stage of preoperational representation do not conceive of the relations in the same way as do older children. Some examples are given of how children at the stage of preoperational representation have difficulty.

Example 1: Establishing the Relation. Imagine that two children drop beans into one jar (Diagram 1) in such a way that for each lima bean one child drops, the other drops a navy bean and the lima beans are exhausted before the navy beans. From the action of placing beans into the jar, the children should know that (1) each child put as many beans in the jar as the other, (2) the child who has navy beans remaining has more beans than the other, and (3) the child who has lima beans has fewer beans than the other. The children should also realize that the navy beans consist of the *remaining navy beans* as well as the *navy beans in the jar*. Five- and six-year-old children (especially the five-year-olds) have difficulty interrelating such knowledge. They may think that the child with navy beans has more beans in the jar than the other child because they may not distinguish the navy beans *in the jar* from those *remaining* and make the comparisons of the beans *in the jar* based on *those remaining*.

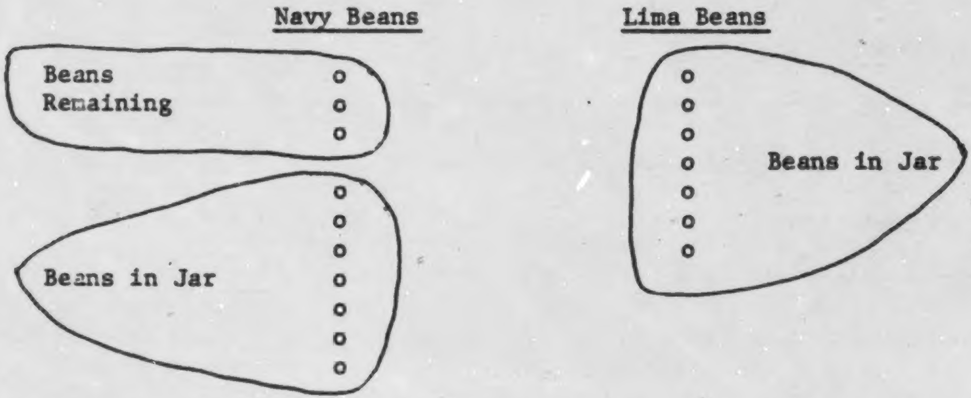


Diagram 1

This example is important in cases where children are being taught to order the whole numbers. At least two distinct varieties of tasks are used in establishing, say, that eight is less than nine and that nine is greater than eight. On the one hand, two distinct collections of objects, one of eight and one of nine, are presented to the child who is required to match the objects of the two collections one-to-one. Based on this matching, the child is expected to realize that eight objects match one-to-one to a subcollection of the nine objects; that there are more objects in a collection of nine objects than in any one of its subcollections; and finally, based on these two reasons, nine objects are more than eight objects. Except for context, this ordering task is analogous to the one presented in the example above. Children at the stage of preoperational representation have difficulty in conceiving of a total collection and two of its subcollections at the same time. If they think of the total collection, they may lose sight of the subcollections, and if they think of the subcollections, they may lose sight of the total collection and make errors as described in the example of placing beans in the jar.

On the other hand, one collection of nine objects is presented to a child who is required to count them. That nine is more than eight is established through noting that eight is how many objects are in a subcollection of nine objects. Here again, the child is espected to realize that any subcollection of a collection of objects has fewer objects than does the original collection.

Example 2: Conserving the Relation. To say that a child should be able to determine that correct matching relations between two collections of objects would be rather artificial if the relation, once it is determined, could not be conserved by the child. As an example, consider the case where a child constructs a matching as pictured in Diagram 2 and says that there are more circles than stars. Then if the stars are altered in full view of the child, as in Diagram 3, and he now thinks that there are more stars than

★ ★ ★ ★ ★

o o o o o o

Diagram 2

circles, it would not be correct to say that he can determine a matching relation between two collections of objects except in a most superficial manner.

* * * * *

oooooo

Diagram 3

Example 3: Effects of Perception. Imagine an experimental setting where two transparent glasses are present of the same size and shape and two piles of beads -- blue beads and red beads. The child and the experimenter place beads in different glasses at the same time -- the experimenter blue beads and the child red beads -- one after the other until the two glasses are full and contain the same number of beads. The child says, "They're both the same." The experimenter and child then take the beads, bead by bead, simultaneously from the two glasses and place them into two other glasses, one taller but narrower than the first glasses and one shorter but wider than the first glasses. After completion, the child is asked whether there are now the same amount in both. He responds, "No, here (the taller glass) there are more because it's big."

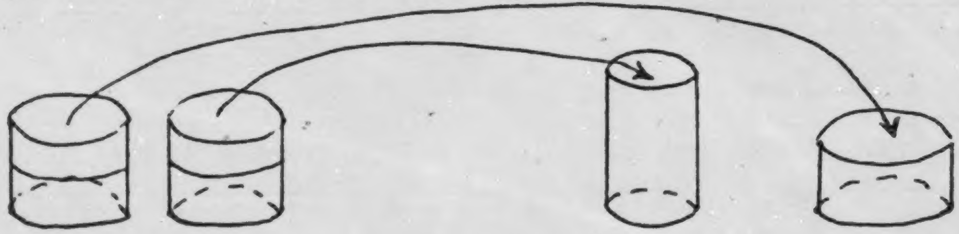


Diagram 4

These responses are quite characteristic of the state of preoperational representation. The perceptual features of the situation completely override the knowledge gained through the correspondence and the correspondence is not lasting in the face of the beads having differing shapes in the two final containers. The relative heights of the blue beads and red beads serve as the basis for the final judgment that there were more blue beads than red beads. One could even argue that actually no correspondence existed for the child when the beads were in the identically shaped containers -- that the initial judgment of equality of the beads was based on the shape and size of the two glasses. The action of placing beads simultaneously into two glasses did not result in a quantifying correspondence for the child. In the experiment, then, the quantitative judgments made by the children could be categorized as gross quantitative judgments. Such judgments are capable of being completely contradictory from an adult's standpoint, although being rational from the child's standpoint.

Example 4: Correspondence in "Transitional" Children. One other experiment with children is cited in order to clarify the role of the unit in one-to-one correspondence. Imagine an experimental setting where the child is asked to place as many (the same number of) candies by a given row of seven candies. Various types of responses are possible on the part of the child in the stage of preoperational representation. Quite often such children will make a row of candies in the row they construct, only placing the correct number of candies in their row purely by accident. These children do not coordinate the length of the row of their candies with the density of the candies in the row, thereby focusing on only one feature (length) of the situation at a time. It would be a mistake to say these children have any conception of one-to-one correspondence or even attempt to construct one -- they react purely on the perceptual features of the configuration of the candies without coordination of those features.

Another level of responses has been identified which go beyond those above. Children may coordinate the length and density of the two rows of candies making two rows of equal length and density but yet not be able to conserve the one-to-one correspondence if one of the two rows is spaced close together or further apart. Such children, however, do believe that if the two rows of candies are identically spaced again, there will be the same number in each. Take the example of the child who made a row of six candies correspond to a given row of six (the model row) by spacing them equally with the model row, but when his row was placed closer together, he then thought there were more in the model row. When asked to make the two rows have the same number of candies again, he spaced his row identical to the model row, not adding nor taking any candies away from either row. This child was definitely more advanced than the children at the stage of preoperational representation (who are only capable of gross quantitative comparison) but yet does not fully comprehend the concept of one-to-one correspondence. His concept is transitional from essentially no concept to an operational concept of one-to-one correspondence. This child is said to be in a transitional stage as regards one-to-one correspondence. The notion of an arithmetical unit is not yet developed, which is essentially what distinguishes an operational one-to-one correspondence from an intuitive one-to-one correspondence (the intuitive one-to-one correspondence is the type children in the transitional stage construct).

Example 5: Operational One-to-One Correspondence. In order to gain a better notion of what is meant by the term "arithmetical unit," it is necessary to distinguish between qualitative one-to-one correspondence and operational one-to-one correspondence. Imagine a situation where there are five toy cars corresponding to five toy garages, one car in each garage -- the red car in the red garage, the blue car in the blue garage, etc. The one-to-one correspondence is based on the color of the cars and garages and is thereby qualitative in nature and is based on the qualities of the elements. But does it go beyond this? It could be an intuitive or an operational one-to-one correspondence -- intuitive if it is not conserved and operational if it is. Being operational means the elements are considered as units -- that is, any car can be placed in any garage -- the color is totally irrelevant to the fact that there are the same number of cars as garages. A given car can be considered to be a placeholder for any other car -- they can be exchanged without a loss in the one-to-one correspondence. An operational one-to-one correspondence demands that the conception of a unit be developed whereas an intuitive correspondence does not. Children capable of establishing operational one-to-one correspondences are capable of making extensive quantitative judgments -- i.e., judgments which are no longer based on only one feature of a situation as are gross quantitative judgments — but where features are coordinated and the notion of an arithmetical unit is developed.

Example 6: Transitivity and One-to-One Correspondence. Knowing that three levels of one-to-one correspondence exist for children -- essentially no one-to-one correspondence, intuitive one-to-one correspondence, and operational one-to-one correspondence -- is important in planning learning activities for children involving one-to-one correspondence. Consider the following problem. Twelve buttons are on a table in front of a cardboard box from which the top and front are removed. A partition divides the box into halves and ten checkers are attached to the bottom of the box on one side of the partition and ten tiles are attached to the bottom of the box on the other side of the partition. The child is asked to find out if there are as many checkers as tiles by using the buttons. The following are protocols of two children who tried the problem.

Deb: (5 years, 11 months).

Experimenter: Deb, find out if there's as many checkers here as tiles here. Use these buttons to find out. (Deb pairs buttons and tiles.)

Deb: Ain't no buttons over there.

Experimenter: No. We just have one pile of buttons.

Deb: Yellow ones is my favorite color -- I got a good idea I can do.

Experimenter: What is that?

Deb: Wait a minute — Put two on each one of 'em.

Experimenter: What are you doing now?

Deb: I had to pick up the two lemon ones.

Experimenter: The two lemon ones?

Deb: Yeah.

Experimenter: What are you doing with the buttons now?

Deb: Pairing them.

Experimenter: Pairing them with what?

Deb: With the tiles and the checkers.

Experimenter: Are there as many checkers as tiles?

Deb: No.

Experimenter: How can you tell that?

Deb: 'Cause these two, these two, these two, these two are missing. Yes.

Experimenter: Yes, there is?

Deb: Yes.

Experimenter: Well now, when you had all the buttons over here except the two lemon ones, were there as many buttons in there as checkers?

Deb: No.

Experimenter: But when you had the two lemon ones here and had all the other buttons in there with the checkers, were there as many buttons as checkers?

Deb: Yes. No.

Experimenter: Let's put them back to see.

Deb: Oh, yes.

Experimenter: All right. Suppose you hand me the two lemon ones and suppose I keep them for you. Now, I wonder if there's as many buttons as tiles?

Deb: Let me see. See my new shoes. You got your Easter shoes?

Experimenter: No, I don't have my Easter shoes.

Deb: I don't have mine neither. It's as many.

Experimenter: There are as many buttons as tiles? (Nod, indicating yes.) Okay, are there as many checkers as tiles?

Deb: I don't know because I can't pair the checkers and the tiles together.

Experimenter: You can't?

Deb: What's this?

Experimenter: That's where your voice is being recorded.

Deb: Can I talk to it?

Experimenter: You're talking to it now -- Did you have as many buttons as tiles?

Deb: Yes.

Experimenter: Okay, how about the checkers and tiles?

Deb: I don't know. (Then Deb points one finger at a checker while pointing another finger at a tile. She then announced:) Yes.

Experimenter: How do you know?

Deb: 'Cuase I point my finger at each of 'em.

Experimenter: I see.

Tom: (6 years, 0 months).

Experimenter: What are these objects, Tom?

Tom: Checkers.

Experimenter: These.

Tom: Tiles.

Experimenter: Tom, I want you to find out if there are as many checkers as tiles. You may use the buttons to find out if there're as many checkers as tiles. Okay, how can you use the buttons?

Tom: Pair 'em --

Experimenter: Tom, tell me what you have done.

Tom: Paired the checkers with the buttons.

Experimenter: What did you find out?

Tom: As many buttons as checkers.

Experimenter: But I want you to find out if there are as many checkers as tiles. Can you do that using the buttons? (Tom pairs buttons and checkers.) Tell me what you have now.

Tom: As many tiles as checkers.

Experimenter: All right, how do you know that?

Tom: 'Cause both of these buttons don't have a checker and a tile for a partner.

Experimenter: All right. There were two buttons left here when we paired them with the checkers?

Tom: Yes.

Experimenter: And there're two buttons left now when they're paired with the tiles?

Tom: Yes.

Experimenter: Does that make it as many tiles as checkers?

Tom: Yes.

Experimenter: Thank you.

Deb never solved the problem of comparing the checkers and tiles. Her comment, "I DON'T KNOW BECAUSE I CAN'T PAIR THE CHECKERS AND TILES TOGETHER" is most revealing -- she can only find out through direct comparison. Her concept of one-to-one correspondence is not operational because the correspondence she established between the buttons and checkers has little significance for her when comparing the two static collections (checkers and tiles). It was as if the first correspondence between the buttons and tiles never existed. Tom's solution was quite sophisticated in that he used the two remaining buttons for his comparison. For him, the one-to-one correspondence established in both cases was related and quantified the sets. One-to-one correspondence for Tom was operational. He therefore was able to use one-to-one correspondence in problem solving, whereas Deb was not able to do so.

In summary, children's conception of one-to-one correspondence passes through three stages -- essentially no one-to-one correspondence, intuitive one-to-one correspondence, and operational one-to-one correspondence. In the two first stages, children do not conserve one-to-one correspondence. However, intuitive one-to-one correspondence is definitely an improvement over essentially no one-to-one correspondence. Children in the stage of preoperational representation are in the first stage of one-to-one correspondence, children in the transitional stage are in the intuitive stage of one-to-one correspondence, and children in the concrete operational stage are in the operational stage of one-to-one correspondence. Children in the operational stage of one-to-one correspondence are able to use one-to-one correspondence in solving problems involving the principle of transitivity (Example 6) and the notion of an arithmetical unit (Examples 4 and 5). In these children the concept of one-to-one correspondence is quite well-developed.

Equivalence and Order Relations

Consider the relation "is a brother of." This relation is meaningful for the collection of living people because it is always possible, given any two people, to determine whether one is a brother of the other. Imagine now that it is known only that A is the brother of B and that B is the brother of C.* What can be concluded about A and C? Try reasoning it out before reading on. A is the brother of B means that A is a male, and B is the brother of C means that B is a male. No information is given regarding the sex of C. But that information is not necessary to conclude for certain that A is indeed the brother of C. The relation "is a sister of" is meaningful for the collection of living people and acts exactly like the relation "is a brother of." That is, through knowing that A is the sister of B and B is the sister of C, then it can be concluded that A is the sister of C. Both relations, then, are transitive relations.

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*It is assumed A, B, and C are different people.

Other relations that do not act like the above two relations are meaningful for the collection of people. The relation "has the same parents as" is a good example. For any two people A and B, the relation is meaningful in that it can always be determined whether A has the same parents as B. Moreover, if A has the same parents as B and B has the same parents as C, then certainly A and C have the same parents. Consequently, the relation "has the same parents as" is said to be transitive, just as are the relations "is a brother of" and "is a sister of." But here, the similarity ends. By knowing only that A is a brother (or sister) of B, it is not possible to conclude for certain that B is a brother (or sister) of A. However, if A has the same parents as B, then it is possible to conclude that B has the same parents as A. The relation "has the same parents as" is then called symmetric, whereas the relation "is a brother of" and "is a sister of" are not symmetric. "Has the same parents as" is also reflexive, because for any person A, A has the same parents as himself (as trivial as it may seem!). A person cannot be his own brother or sister, so these two relations are not reflexive.

Consideration of the relation "has the same parents as" goes a long way toward the identification of a general type of relations. Relations that are reflexive, symmetric, and transitive are called relations of equivalence, or equivalence relations. These relations separate the collections on which they are defined into subcollections. In case of "the same parents as," the set of living people are separated into collections called "families," where a "family" is a collection of full brothers and sisters. It is quite possible for some collections to contain only one member -- which is why it is important that a person be considered as having the same parents as himself. The collection of living people, then, may be classified into subcollections (called "families") on the basis of the relation "has the same parents as." This classification is exhaustive, which means that every person belongs to some family. Moreover, the classification has the property that no person belongs to two "families" -- each person belongs to one and only one "family."

Other relations exist, meaningful for the set of people, which determine collections in the same way as does "has the same parents as." One such relation is "same height as." For any two people A and B, if it is found that A is the same height as B, then it can be concluded that B is the same height as A (symmetric). Moreover, any person is the same height as himself (reflexive). Also, if A is the same height as B and B is the same height as C, then A is the same height as C (transitive). These three properties are quite necessary for a relation to be called a relation of equivalence, because in the absence of any one of them, the relation cannot determine a classification. The reader should attempt to determine whether the relations "the same weight as," "the same grandparents as," "the same number of arms as," "the same eye color as," "speak the same languages," and "is the uncle of," considered for the collection of living people, are relations of equivalence and attempt to identify other relations of equivalence.

Relations of equivalence are not the only types of relations of importance. Take the example "shorter than." This relation is meaningful on a collection C of people (no two of which are the same height) because for any two such people A and B it can be determined whether A is shorter than B. The collection C can be ordered from shortest to tallest using the relation "shorter than" in the following way. First, the shortest person is identified (call him P_1), then the shortest of those remaining is identified (call him P_2), then the shortest of those remaining is identified (call him P_3), then etc., until the last person, the tallest, is identified. Then, if "<" is read "shorter than," the persons of collection C have been ordered from shortest to tallest where the ordering is represented as follows: $P_1<P_2<P_3<P_4<P_5. . .$, it is to be read, "The first person is shorter than the second, the second is shorter than the third, the third is shorter than the fourth, etc. The three dots indicate that the process continues until the last person (the tallest) is reached.

The relation "shorter than" can be also defined for a collection of strings in the same way as it was for the collection of people. A given collection of strings may be ordered from shortest to longest through comparison activities. In such comparison activities, the following principles are involved. If, through direct comparison, it is found that string A is shorter than string B and string B is shorter than string C, it may be concluded, without comparison, that string A is shorter than string C. This principle is called transitivity of "shorter than" and is analogous to the transitive principle of other relations discussed. Another principle of important is that if it is known that string 1 is shorter than string 2, then string 2 is not shorter than string 1 (in fact, it is longer than string 1). This principle is called the asymmetric principle of "shorter than."

To seriate a collection of sticks means to place them in order either from the shortest to longest, or from the longest to the shortest. But, for a given collection of sticks, it is quite possible that some sticks are the same length. To seriate such a collection, it would be necessary to classify the sticks together that are the same length and then seriate these classes For example, consider a collection of 20 sticks; four, five inches long; three, six inches long; one, eight inches long; five, nine inches long; four, ten inches long; and three, eleven inches long. To seriate this collection, ultimately all of the sticks the same length must be classified together. Let C be the four sticks five inches long, C_2 the three sticks six inches long, C_3 the one stick eight inches long, C_4 the five sticks nine inches long, C_5 the four sticks ten inches long, and C_6 the three sticks eleven inches long. Then $C_1<C_2<C_3<C_4<C_5<C_6$, where "<" means any stick of C_1 is shorter than any stick of C_2, and so forth. Sets of objects can be seriated just as well as sticks. The order relation is "fewer than," of course, rather than "shorter than," (or "more than" rather than "longer than").

Children's Conception of Equivalence and Order Relations

Equivalence Relations. It has been noted earlier that equivalence relations are special kinds of relations. One of the special characteristics of equivalence relations is that an equivalence relation separates the elements of the collection on which it is defined into subcollections. An example given was "has the same parents as" defined on the collection of living people. This relation determines subcollections ("families" of full brothers and sisters) of people where any one person does not belong to two families but does belong to some family. Another example is "the same length as" defined on a collection of sticks.

If a child is given a collection of sticks and asked to put the sticks into piles so that the sticks in any one pile are the same length, he must base his classification activities on the relation "the same length as." It would seem essential for the child to employ properties (reflexive, symmetric, and transitive) of this equivalance relation. In order to clarify this, the following analysis is given of a child's behavior in sorting a collection of sticks into piles.

The child must select some stick, say stick s, from the given collection of sticks and hunt for another stick r the same length as s. If no such stick r exists, then the child must classify s with itself (reflexive property). If some stick r does exist the same length as s, then the child must realize that not only is s the same length as r, but r is also the same lenth as s (symmetric property) for r and s to be considered as forming a class. Given that r and s are classified together, then the child at some time must hunt through the sticks yet to be classified to determine if there is another stick the same length as s (and r). Suppose that such a stick t exists. For t to be classified with r and s, the child must realize that all three are the same length, which entails knowledge of transitivity of "same length as." (s is the same length as t and r is the same length as s, so r and t are the same length.) One may object and say that all the child has to do is compare t with r and s by placing the sticks together, so transitivity has never to be used! Our contention is that children would not think of classifying the sticks together in absence of the ability to employ transitivity, the reflexive property, and the symmetric property.

In order to obtain some idea of what to expect from first and second grade children regarding their classification activities based on "the same length as," 81 first and second grade children (39 first and 42 second) were given three tasks to perform. In the first task, the children were given a collection of sticks to sort into three collections and three sticks on which to base the sorting. Each of the sticks in the collection was exactly as long as one of the three sticks. Sixty-two of the children correctly sorted the sticks into three piles.

In the second task, the children were given another collection of sticks to sort but were not given three sticks on which the base the sorting. Here, the children were asked to put all of the sticks together that belonged together. Only six of the 81 children did not attempt the task. However, only 37 children completed the second task as compared to 62 in the first task. The 44 children who did not complete the task made some piles, but did not connect together piles that went together and generally placed sticks into piles incorrectly.

The third task consisted of giving the children three piles of sticks already sorted together on the basis of "same length as" and asking them why the sticks were put together in the way they were. Fifty-nine children did not discover the basis for the classification (same length as). The remaining 22 children showed some evidence of being aware of the classification.

Classification. Classification has been studied more widely than just in relation to "the same length as." In order to fully appreciate classification behavior of children, it is necessary to discuss classes (or sets) per se. Generally, when objects are classified together, they share common properties. For example, quite dissimilar objects can be classified together under the heading, "fruit." What makes these objects "fruit" but not "vegetables" is what is common. Within the class of fruit, however, important differences exist -- oranges and apples are different. So, given a universe of objects -- in this case edible items by humans -- two distinct kinds of properties or relations exist.

1. (a) Properties specific to members of a given class (e.g., the properties which make edible items fruit) which distinguishes the class from other classes (from vegetables, meat, etc.).

(b) Properties which are common to members of a given class and those of other classes to which it belongs (that which is common to fruit and other edible items).

(c) Properties which differentiate members of a given class one from another (those which differentiate a pear from an apple, for example).

2. Part-whole relations of class membership and inclusion. These relations are conveyed by the terms "all," "some," "one," "none," when applied to the members of a given class and those of the classes to which it belongs (all oranges are fruit, some fruit are apples, no fruit are vegetables).

The difference in (1) and (2) above lies in what is known as the intension of a class and the extension of a class. The intension of a class is the properties common to the elements, and the extension of a class is just the members of the class. The coordination of the intension and the extension of a class is what develops in children in stages. These stages correspond to the three stages identified in the development of one-to-one correspondence.

Very young children (below about six years of age) employ primitive behavior in attempting to form classifications. The types of collections formed by these children have been called complexive collections (or graphic collections). For example, children were asked to classify a collection of geometric objects together, some triangular shapes, some square shapes, and some half ring shapes. At least three varieties of graphic collections were identified. First was where children constructed a number of subcollections, ignoring the rest of the material which was never classified. The subcollections had no common property -- the child would change criteria of classification within a subcollection. At times, subcollections would not be formed, but properties of individual items noted.

Second is where successive similarities between one object and the next was formed. While this is an improvement over the type of behavior noted in the first example, it is not true classification as no over-all criteria for classification was found for subcollections; subcollections were not differentiated, and part-whole relationships were not identified.

Third is where definite figures are made out of the objects -- a "house" is made, then windows, etc. That is, the child makes no real attempt at classification, but instead plays with the objects, making whatever comes to his fancy.

The graphic collections described above have two essential aspects which make them different than true classes. First, the collections are formed on the basis of the spatial arrangement of the objects. Second, no criteria for classification (no properties which tied the elements together) were isolated by the children. These two aspects are simply another way of saying that intensive properties were not identified by the children -- these children are at Stage I (preoperational) as regards their classification behavior.

Stage II (or transitional) classification behavior is an advance over Stage I classification behavior, but it is not yet operational classification behavior. Stage II classification behavior can best be characterized by a recognition of intensive properties, but yet no coordination between the intension of a class and the extension of a class exists for the child. More precisely, children, given a class of objects, are able to separate the class of objects into subclasses. This means that they understand that all elements can be classified, each subclass contains elements of a specific kind or which possess a specific property, and two or more subclasses are constructed. Yet, the subclasses formed are not thought of as forming an hierarchy of classes, which is another way of saying that the class-inclusion relation is not mastered.

The class-inclusion relation being mastered means simply that, given a class A which is contained in a class B, so that all of the A are some of the B, but all of the A do not constitute all of the B, the child understands both of the statements. For example, if A is the class of Siamese cats and B is the class of cats, then all Siamese cats are certainly cats, but they do not exhaust the cats. That is, there are cats that are not Siamese cats. So, all of the A do not constitute all of the B, but just some of the B. Children at the transitional stage of classification certainly realize that Siamese cats are indeed cats, and in fact are part of the cats. So, one would think they would understand class-inclusion. But they do not. It is critical they understand that there are more cats than Siamese cats or, in other words, that all cats are not Siamese. If A' are the non-Siamese cats, then $A \cup A' = B$ and $A = B - A'$ (see diagram).

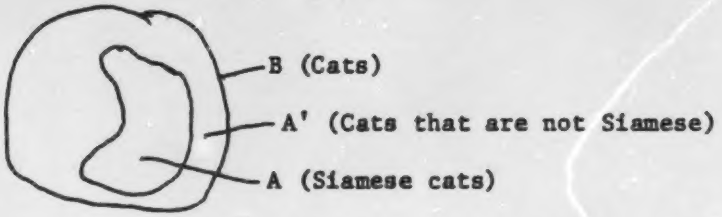


Diagram 5

To understand class-inclusion, the child must be able to engage in reversible thinking. To do so is to be able to conceive that the Siamese cats, together with the non-Siamese cats ($A \cup A'$), make up the cats (B); and that the cats, minus the non-Siamese cats, make up the Siamese cats ($A = B - A'$). In this reversible reasoning, the child has to be able to conceive of the total class of cats as being made up of the *two* subclasses at one and the same time. Stage II children, when focusing on the cats, lose sight of the subclasses, and when focusing on the subclasses, lose sight of the total collection. Typical responses of transitional children (Stage II) are given in the following situations. A picture is shown to the children on which there are, say, four Siamese cats and three cats which are not Siamese. The children are asked to compare the number of cats to the number of Siamese cats. When asked to do so, the children will compare the Siamese cats to the other cats. In situations such as the one given in Diagram 6, if a Stage II child is asked whether there are more shapes or more squares, he will compare the squares and circles and respond "more squares."

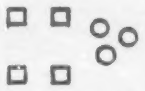


Diagram 6

The children at Stage III (concrete operational) are capable of solving the class inclusion problem, and are much more flexible in their classification behavior than are Stage II children. Stage II children are able to build hierarchies of classes. For example, they are capable of conceptualizing such hierarchies as maltese terriers are part of the terriers, terriers are part of the dogs, dogs are part of the mammals, etc. Stage III children are not only capable of building hierarchies of classes, but are able to change the criteria of classification once it is determined and classify a set of elements a whole new way. In the example immediately above, the child may consider new dogs in his classification and refine the classification to include many more classifications than those given. So, basically, two complementary processes exist which describe the Stage III flexibility in classification. One is that given a classification is constructed, the child can go back and construct finer classifications or whole new classifications and not be tied to the one constructed. The other is that a child can anticipate a classification before it actually is carried out.

In summary, the following three stages in children's classificatory behavior have been identified:

> Stage One. (Preoperational) When given a collection of objects and told to "put everything together that goes together," a child at this stage forms what is known as "graphic collections," that is, he constructs one or more spatial wholes. This is a child's first attempt to coordinate part-whole relations with those of equivalence and difference.
>
> Stage Two. (Transitional) At this stage, the constructed collections are no longer graphic collections. Trial and error plays a large role in construction of classifications and no over-all plan is present. Children cannot yet solve the class-inclusion problem but do understand that all elements need classifying, each subclass contains elements which possess a specific property, and two or more subclasses are constructed.
>
> Stage Three. (Concrete Operational) Children at this stage are able to coordinate the intension and extension of a class, as evidenced by the solution of the class inclusion problem. Children at this stage are capable of conceiving of hierarchical arrangements of classes, and are capable of imposing more than one classificational system on the same collection of elements, anticipating the new classificational systems before carrying out the classification.

Order Relations and Seriation

Order relations determine a seriation of the objects on which they are defined just as equivalence relations determine a classification of the objects on which they are defined. Three stages exist in the development of seriation behavior.

Stage I. (Preoperational) This stage is characterized by no attempt at seriation or the forming of small uncoordinated series.

Stage II. (Transitional) This stage is characterized by seriation by trial and error.

Stage III. (Concrete Operational) This stage is characterized by a systematic method of seriation.

At Stage I, the child either does not attempt to form a series or else just forms small uncoordinated series of two or more elements. In the latter case, the subseries are not connected for the child (ı| ı| ıı|). The representation in parentheses is supposed to connote that the child first orders two sticks (any two), then two more, then three more, never realizing that the sticks need to be ordered into one series.

At Stage II, the child is not systematic. He can form a series, but does so with no overall plan or complete anticipation of what he is to do. For example, a child may pick two sticks and put them in order, pick two more and then put them in order, and then attempt to coordinate the four sticks into an order, etc. Or, a child may lay a whole "series" out and then attempt to put them in order through a process of trial and error. This is definitely an advance over Stage I seriation behavior.

At Stage III, children proceed systematically, choosing the smallest element (or largest, depending on where they start), then the next smallest, etc., until they are done. These children know beforehand that a given stick (say the third) is going to be longer than those already chosen, but shorter than all those yet to be chosen. Children at the second stage do not realize this, being capable of thinking in one direction only.

NUMBER

Number has yet to be explicitly mentioned in this chapter. Relations have been presented as a mathematical topic and as a topic of cognitive development. It was shown that equivalence relations are inherently connected up with classes and order relations are inherently connected up with series, as mathematical topics. It was also shown that classification and seriation behavior of children are closely allied with equivalence and order relations, respectively. That is, the growth in the child of an ability to classify and seriate in closely paralleled with the growth of an ability to engage in relational thinking.

Attention is now turned toward the topic of cardinal and ordinal number. In the mathematical development, it will become clear that relations are at the heart of number. The child's conception of number, then, will be shown to closely parallel his conception of relations, and thereby his ability to classify and seriate.

Set Equivalence

Two sets of objects are said to be equivalent whenever there is a one-to-one correspondence between the elements of the two sets. If $A = \{a, b, c, d, e\}$ and $B = \{\alpha, \beta, \gamma, \varepsilon, \sigma\}$, then A and B are equivalent because a one-to-one correspondence can be established between the elements of A and B. One such correspondence is displayed in Diagram 7. Obviously, more than one such

$a \leftrightarrow \alpha$
$b \leftrightarrow \beta$
$c \leftrightarrow \gamma$
$d \leftrightarrow \varepsilon$
$e \leftrightarrow \sigma$

Diagram 7

correspondence exists. In fact, 120 different correspondences exist between two five element sets. But it makes no difference which correspondence is set up to establish that A and B are equivalent, just as long as one is displayed.

If C is another collection of objects where C = {r, s, t, u, v}, then a one-to-one correspondence may be established between the elements of B and C. From knowing that a and α are paired in the first correspondence, then a and r are naturally paired and a one-to-one correspondence between A and C exists.

$\alpha \leftrightarrow r$
$\beta \leftrightarrow s$
$\gamma \leftrightarrow t$
$\varepsilon \leftrightarrow u$
$\sigma \leftrightarrow v$

Diagram 8

Now, in the examples given, it is quite obvious that A and C are equivalent. But what if three sets A, B, and C exist, where each contains a rather large number of elements (greater than 50). If it is known that A and B are equivalent and that B and C are equivalent, it is possible to infer that A and C are equivalent without knowing how many objects are in each. The basis for the reasoning is just that the one-to-one correspondence which exists determines a pairing between the elements of A and C. So, set equivalence is a transitive relation. A given set is also equivalent to itself (reflexive property) and if set A is equivalent is set B, B is also equivalent to A (symmetric property). So, set equivalence is an example of an equivalence relation. It is important for the reader to realize that set equivalence is a relation between sets of objects and is based on one-to-one correspondence.

Cardinal Number

Given a set A, an object a can be associated with A, where a is called the cardinal number of A. If A = {a, b, c, d, e}, then the usual object "5" is associated with A. Note that nothing has been said about the object "5" except that it is an object associated with a set. Remember that B = {α, β, γ, ϵ, σ} is in one-to-one correspondence with A. The cardinal number of B, then (which is just an object at this point), is said to be equal to the cardinal number of A because the elements of A and B are in one-to-one correspondence. If b is the cardinal number of B, then b = 5, where 5 is the cardinal number of A. This leads us to the following definition: If A and B are equivalent sets, then the cardinal number of A is equal to the cardinal number of B. (Also, if the cardinal number of A is equal to the cardinal number of B, then A and B are equivalent sets.)

Set equivalence, being an example of an equivalence relation, determines classes of sets, where any two sets in a given class are equivalent. For example, "4" is associated with the set {a, b, c, d} and also with a whole host of other sets, each in one-to-one correspondence with {a, b, c, d}.

In mathematics a cardinal number, such as 4, is then considered as being a class of sets, where the sets are all equivalent. At first, this may seem strange, but certainly it just as meaningful as how the word "cat" is used. A child's first encounter with a cat may be a black cat. Then the word "cat" may be associated with a Persian, then a calico, etc., until he has associated the word with a variety of cats. For the word "cat" to refer to all cats, the child must have recognized the properties of cats that make cats "cats," or in other words, uses the word to refer to particular cats and also the class of cats. So, there is a member-of-a-class meaning of the word "cat" and a class meaning. The same is true of "four" (or any other such number). The member-of-a-class meaning of "four" is when "four" is used to refer to the legs of a horse, the legs of a chair, etc. The class meaning of "four" is when "four" is used to refer to a whole host (class) of sets in one-to-one correspondence with {a, b, c, d}.

The cardinal numbers of two sets not in one-to-one correspondence may be compared in the following way. If A and B are two sets of objects where A is not equivalent to B, then either the elements of A are in one-to-one correspondence with a subset of elements of B or vice versa. In the former case, we will say there are fewer elements in A than in B and that the cardinal number of A is less than the cardinal number of B.

For example, if A is a collection of jelly beans and B a collection of children, where there are not enough jelly beans for each child to have exactly one, then there are fewer jelly beans than children and more children than jelly beans -- there are not enough jelly beans for each child to have one! The number of jelly beans is less than the number of children. Note it is not necessary to know how many jelly beans or children there are to make the conclusion that the number of jelly beans is less than the number of children. Moreover, if a sack of suckers exist, where the suckers are in one-to-one correspondence with the jelly beans, then it can be concluded that there are fewer suckers than children -- not enough suckers! If there is another class of children in one-to-one correspondence with the original class, then it can be also concluded that there are fewer suckers than children in the second class, without ever actually setting up a one-to-one correspondence (by giving each child a sucker).

By using the relation "fewer than," a collection of nonequivalent sets may be ordered (or seriated) in the same way as a collection of sticks may be seriated. If A, B, C, D, E, F, etc. are the sets and "α" the relation "fewer than," then the ordering $A \alpha B \alpha C \alpha D \alpha E \alpha F \alpha$, etc. may be accomplished (it is assumed the relation holds in the given ways). If a, b, c, d, e, f, etc. are the cardinal numbers of the sets, then the cardinal numbers may be ordered by "less than" based on the ordering "fewer than" of the sets. So, $a < b < c < d < e < f <$, etc. is an ordering of the cardinal numbers of the sets.

Some of the important principles must be highlighted at this time. One is: if a is the cardinal number of A and A is equivalent to some other set A_1, then a is also the cardinal number of A_1. A second is that if $A \alpha B$ and A is equivalent to A_1 and B is equivalent to B_1, then (1) $A_1 \alpha B_1$, (2) $A \alpha B_1$, and (3) $A_1 \alpha B_1$. A third is that if $A \alpha B$ and $B \alpha C$, then $A \alpha C$. This third principle is nothing more than transitivity of "fewer than" -- if it is known that there are fewer objects in A than in B and fewer objects in B than in C, then it can be concluded that there are fewer objects in A than in C.

By a particular choice of the sets A, B, C, D, E, F, etc., it is possible to arrange the cardinal numbers in their natural order. If $A = \{n\}$, $B = \{\beta, \gamma\}$, $C = \{\delta, \sigma, \lambda\}$, $D = \{\varepsilon, \pi, \theta, r\}$, $E = \{\rho, t, u, v, w\}$, $F = \{r, s, m, x, y, z\}$, etc., then an ordering is possible. Let n correspond to β; β and γ correspond to δ and σ; δ, σ, and λ correspond to ε, π, and θ; ε, π, θ, and r correspond to ρ, t, u, and v; and ρ, t, u, v, and w correspond to r, s, m, x, and y. All of these correspondences are represented in Diagram 9. Thus,

$$\begin{array}{ccccccccccc}
1 & < & 2 & < & 3 & < & 4 & < & 5 & < & 6\ldots \\
A & \alpha & B & \alpha & C & \alpha & D & \alpha & E & \alpha & F\ldots \\ \hline
n & \leftrightarrow & \beta & \leftrightarrow & \delta & \leftrightarrow & \varepsilon & \leftrightarrow & \rho & \leftrightarrow & r \\
 & & \gamma & \leftrightarrow & \sigma & \leftrightarrow & \pi & \leftrightarrow & t & \leftrightarrow & s \\
 & & & & \lambda & \leftrightarrow & \theta & \leftrightarrow & u & \leftrightarrow & m \\
 & & & & & & r & \leftrightarrow & v & \leftrightarrow & x \\
 & & & & & & & & w & \leftrightarrow & y \\
 & & & & & & & & & & z
\end{array}$$

Diagram 9

n ↔ β means n corresponds to β, and so forth. The elements of each set are listed below the letter for each set. For example, ε, π, θ, and r are all listed directly below D, the set which contains them. B α C means that there are fewer elements in B than in C and this is illustrated by the correspondences between the elements shown directly below B α C. A contains one element, B contains two elements, etc., so that the ordering "fewer than" determines the ordering "less than" between the corresponding cardinal numbers of the sets. By virtue of this ordering, we can say that 1 is less than 2 and 2 is greater than 1; 2 is less than 3 and 3 is greater than 2; 3 is less than 4 and 4 is greater than 3; etc. The "one more than" relation is also clear. That is, 2 is one more than 1; 3 is one more than 2; 4 is one more than 3; 5 is one more than 4; 6 is one more than 5, etc.

Ordinal Number

The processes involved in rational counting have not yet been fully explained. One would think that arranging the cardinal numbers in their natural order as above would be enough to account for counting. When one counts, however, a single set of objects is counted. In the example above, each number was associated with a different set of objects. While certain restrictions can be placed on the sets so counting is possible, it is simpler to look at counting, and then explain what these restrictions would have to be.

Rational counting is based on ordinal number. In the discussion of number up to this point, cardinal numbers were being discussed. The ordering of the cardinal numbers above should not be mistaken for ordinal numbers, even though ordinal numbers are closely associated with order. Given a set A, where A = {a, b, c}, the elements of A can be arranged in the ways shown in Diagram 10.

1. a, b, c
2. a, c, b
3. b, a, c
4. b, c, a
5. c, a, b
6. c, b, a

Diagram 10

In each one of these arrangements, the symbols "a," "b," and "c" are arranged in a different order. So, six ordered sets are determined, one for each arrangement. The ordered sets are: {a, b, c}; {a, c, b}; {b, a, c}; {b, c, a}; {c, a, b}; {c, b, a}. Any two of these sets are not equal ordered sets, but are equal in the sense that they have the same elements. The set {a, b, c} and the set {b, c, a} are not ordered the same way, so are not equal ordered sets, but are equal sets if the order of the elements is not considered.

Given the ordered set {a, b, c, d, e, f}, each element determines part of the set in the following way. If any two elements were interchanged, then a different ordered set would be determined. So, in an ordered set, any element is in a certain place and cannot be interchanged and still have the same ordered set. The position of each element, then, is important in that each element determines a subset of the original set. This subset is just itself and the elements preceding it. For example, in the ordered set {a, b, c, d, e, f}, "c" is in the position it is because the elements "a" and "b" precede it. So, "c" determines the ordered subset {a, b, c} of the ordered set. The element "e" determines the subset {a, b, c, d, e}, itself an ordered set. All the elements of the set (except e and f) precedes "e," and "e" is said to succeed {a, b, c, d}. So, "a" precedes all other elements of the set {a, b, c, d, e, f} and does not follow any other element. It determines the subset {a}. The element "b" follows the element "a" and precedes the remaining elements. It determines the ordered subset {a, b}, which is called the successor of {a}. The element "c" follows the element "b" and precedes the remaining elements "d," "e," and "f" and determines the ordered subset {a, b, c}, the successor of {a, b}. The element "d" follows the element "c," precedes "e" and "f" and determines the ordered subset {a, b, c, d}, the successor of {a, b, c}. Of the two remaining elements, "e" follows "d," precedes "f," and determines the ordered subset {a, b, c, d, e}, the successor of {a, b, c, d}. The element "f" follows "e" and determines the whole ordered set {a, b, c, d, e, f}.

Each ordered set has an object called an ordinal number associated with it. Thus, the ordered set {a, b, c} has the ordinal number "3" associated with it. This immediately raises the question, "what is the ordinal number of {b, c, a}?" It is, of course, also "3." But what is the justification for the assignment of the same ordinal number to each of the two ordered sets? We saw earlier that {a, b, c} is not equal to {b, c, a} because they are ordered differently. So, why do they have the same ordinal number? To answer this last question, consider the following two sets. The first is a set of 10 dolls ordered from shortest to tallest (assuming no two are the same height) and the second is a set of 10 hats ordered from smallest to largest, according to size. These two sets can be placed into one-to-one correspondence because each contains 10 members. However, there is only one correspondence which preserves the order of the two sets. That correspondence is where the smallest hat is given to the shortest doll, the next smallest hat is given to the next shortest dolls, etc., until the largest hat is given to the tallest doll. Diagram 11 contains a pictorial representation of this one-to-one correspondence. Note that given any two hats and their corresponding dolls, by knowing which hat is smaller, it is known which doll is shorter, and vice versa.

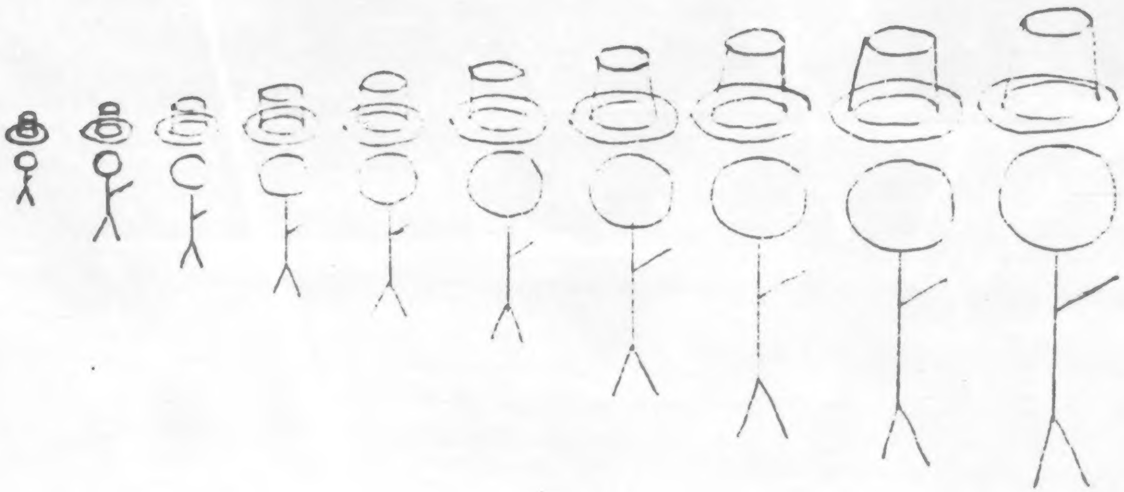


Diagram 11

It is important to realize that while the dolls can be ordered from shortest to tallest, any arrangement of the dolls constitutes an ordering of the dolls (but not necessarily an ordering by height). The ordering is one of position in a series of dolls. Let's say we want to count the dolls. Any one of the ten could be chosen first. Then any one of the nine remaining could be chosen as second, etc., until the last one is chosen. The number of possible arrangements is 10·9·8·7·6·5·4·3·2·1, or 3, 628,800. But, no matter how the dolls are selected, there are ten! So, given the set {a, b, c} and the set {b, c, a}, a one-to-one correspondence can be set up as in Diagram 12. This one-to-one correspondence preserves the order of the two sets because in the first set, a precedes c; a (in the first set) corresponds

$$\begin{array}{ccc} a & b & c \\ \updownarrow & \updownarrow & \updownarrow \\ b & c & a \end{array}$$

Diagram 12

to b (in the second set) and c (in the first set) corresponds to a (in the second set); and b precedes a in the second set. No matter which two elements in the first set you take, you can determine which of their corresponding elements will precede the other by knowing which of the two original elements comes first. So, the correspondence is said to preserve the order of the two original sets.

Another example of a one-to-one correspondence which preserves the order of the two original sets is Diagram 11. There, the correspondence preserved the order of the hats from smallest to largest and the height of the dolls from shortest to longest. No matter which two hats you take, you can order the corresponding dolls by knowing just which of the hats is smaller. But, we have noted already that the hats can be ordered by position rather than by size, and that the dolls can be ordered by position rather than by height. Any arrangement of the hats, then, can be placed into one-to-one correspondence with any arrangement of the dolls, where the one-to-one correspondence preserves the order of position. That is, the hat in the sixth position corresponds to the doll in the sixth position (whatever they may be) and if another hat is in some other position, say the fourth, it corresponds to the particular doll in the fourth position.

Assume now that there is a universe of elements from which we can choose singular elements at will. The nature of the elements are of little consequence -- they can be any sort of objects as long as they are distinguishable. Imagine that an element is chosen, say "a." The set {a} has the ordinal (and cardinal) number "one" assigned. Another element, say "b," is chosen, and the successor set {a, b} is formed. This set is an ordered set by the way the elements were chosen. The element "b" comes after the element "a" in the act of choosing. The ordered set {a, b} has the ordinal number "two" assigned. Any other set in one-to-one correspondence with {a, b} also has "two" assigned as its ordinal number. In fact, "two" is regarded as the cardinal number of {a, b} and ordinal and cardinal numbers have the same names. The ordinal number "two" contains more information than the cardinal number "two," because one and only one element (in this case, b) is the second element. The cardinal number "two" does not tell which element is second. In each case, "two" is considered as referring to a whole assemblage of sets in one-to-one correspondence with {a, b}. When we are considering the ordinal number "two," the one-to-one correspondence includes order, and when we are considering the cardinal number "two," the one-to-one correspondence does not include order.

Now, consider that another element is chosen, say c, and the successor set to {a, b} is formed -- namely, {a, b, c}. This set is an ordered set with the ordinal number "three" assigned, which is also the cardinal number of the set. The element "c" is the third element chosen, "b" is the second element, and "a" is the first element. The ordinal number "three" is considered as referring to a whole assemblage of sets each in one-to-one correspondence with the ordered set {a, b, c} and the cardinal number "three" is considered as referring to a whole assemblage of sets each in one-to-one correspondence with the set {a, b, c}, not considering order. Again, the ordinal number "three" carries more information than does the cardinal number "three" in that the ordinal number tells which element is third, which is second, and which is first as well as how many elements there are. The cardinal number "three" tells just how many there are.

Ordinal numbers are used in counting. Given a set of elements to count, the first element "a" is selected and the set {a} assigned the ordinal number "one," an element "b" is then selected and the successor set {a, b} is given the ordinal number "two," an element "c" is selected and the successor set {a, b, c} is given the ordinal number "three," an element "d" is selected and the successor set {a, b, c, d} is given the ordinal number "four," and so on, until the last element is selected. The original set to be counted can now be represented as {a, b, c, d, . . . , θ}, where the three dots mean that some elements have not been represented, θ is the last element, and the number of elements is the ordinal number of the set (whatever it is!). If the set had 26 elements, the alphabet is represented. Diagram 13 contains a representation of the counting process in the case where there are nine elements. The symbol $\circledast^{1}$ means one element has been selected (it is *);

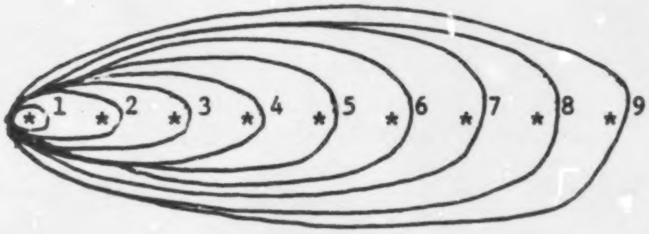


Diagram 13

$\circledast^{2}$ means two elements have been selected, where the 2 tells how many elements have been selected and that the second one is the * outside the first ring, etc. up until all nine elements have been counted.

It is convenient to have what is known as the counting set identified. The counting set is just the set of ordinal numbers {1, 2, 3, 4, ... , n}, where n can be any ordinal number. This set of numbers is used for counting sets of objects as in Diagram 13. Anyone who counts in the way described in that diagram is said to count rationally.

Now, referring back to Diagram 9, it is easy to see what restrictions would have to be placed on the elements of the sets in order to have represented counting of a set. The element n and the element β would have to be the same element. So, we will replace β by n. In fact, σ, ε, ρ, and r will all be replaced by n (the top row). The second row of elements will be replaced by γ; the third row by λ; the fourth row by r; and the fifth row by w.

The diagram then is represented as in Diagram 14. In this case, 1, 2, 3, 4, 5 and 6 represent ordinal numbers as well as cardinal numbers. Diagram 15

$$\begin{array}{cccccc}
1 < & 2 < & 3 < & 4 < & 5 < & 6 \ldots \\
\hline
A \, \alpha & B \, \alpha & C \, \alpha & D \, \alpha & E \, \alpha & F \ldots \\
n \leftrightarrow & n \leftrightarrow & n \leftrightarrow & n \leftrightarrow & n \leftrightarrow & n \\
 & \gamma \leftrightarrow & \gamma \leftrightarrow & \gamma \leftrightarrow & \gamma \leftrightarrow & \gamma \\
 & & \lambda \leftrightarrow & \lambda \leftrightarrow & \lambda \leftrightarrow & \lambda \\
 & & & r \leftrightarrow & r \leftrightarrow & r \\
 & & & & w \leftrightarrow & w \\
 & & & & & z
\end{array}$$

Diagram 14

is much simpler and illustrates that 2 is one more than 1; 3 is one more than 2; 4 is one more than 3; 5 is one more than 4; and 6 is one more than 5.

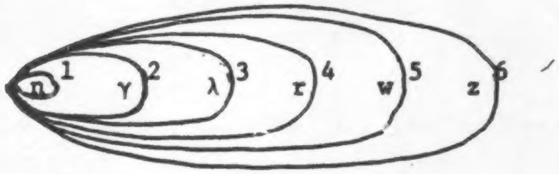


Diagram 15

Numeration

It is convenient to think of a cardinal (or ordinal) number as a representative set. For example, three will be thought of as {a, b, c} or any set equivalent to it -- {a, b, c} is used as a representative set for three. In the following discussion, it should be clear from the context whether cardinal numbers, ordinal numbers, or both are being referred to. Whenever clarification is needed, reference to one or the other will be made.

Given a large collection of objects to count, it is not necessary to employ the concept of place-value to name the numbers if the written (or spoken) number names are known in sequence and rational counting is possible on the part of the one who is doing the counting. Indeed, children learn the number names in sequence long before they ever encounter place-value concepts. In order to count, the counting set may be represented as (one, two, three, four, five, six, seven, eight, nine, ten, eleven, twelve, thirteen, . . . , n), where n is the name for some ordinal number. If the collection of objects happen to be a collection of blocks, then the collection may be counted to find how many blocks the collection contains, as diagrammed in Diagram 16. In the diagram, the layout means that one block is selected; then another is put with the first two, so three are present; then another is put with the first three, so four are present, and so on; until the last one is selected. If there happened to be 36 blocks, the name "thirty-six" may be spoken to refer to an ordered collection of thirty-six blocks, ordered by successive selection of blocks. The "thirty-six" names the same number that "36" does, but "36" usually is taken to mean that there are three tens and six ones -- that the blocks have been portioned into three sets of ten objects per set and six sets of one per set.

□	□□	□□□	□□□□	□□□□□	□□□□□□	
one	two	three	four	five	six	, etc.

Diagram 16

Realizing that a collection of objects can be counted without using the concept of place-value is quite important, especially when dealing with young children. Counting by ones and using number names to refer to particular sets of objects which have been counted is quite acceptable mathematical behavior, even though the concept of place-value is one of the most important concepts in elementary school mathematics. It must, however, have rational counting as a foundation.

The ideas contained in the decimal system of numeration are quite simple. Only ten symbols, called digits, exist. They are "0," "1," "2," "3," "4," "5," "6," "7," "8," and "9." The symbols used to represent all other ordinal (and cardinal) numbers are just combinations of these symbols. Imagine now, that we have a collection of elements to count. Following the usual procedures of counting, a first is chosen, then a second, then a third, etc. The particular collection named by "10" is important because, after selecting 10 elements, we continue counting single elements, holding in mind the collection of 10 already counted. Ignoring the number names for numbers greater than 10, such a counting procedure is characterized as follows (the number names are to be spoken).

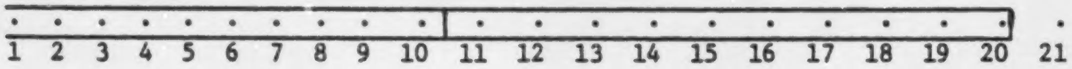


Diagram 17

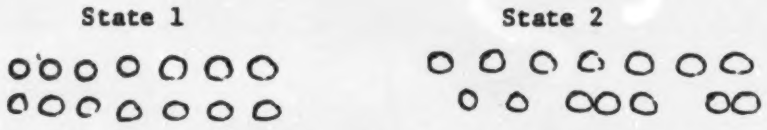
> one, two, three, four, five, six, seven, eight, nine, ten
> one ten and one, one ten and two, one ten and three,
> one ten and four, one ten and five, one ten and six, one ten
> and seven, one ten and eight, one ten and nine, two tens,
> two tens and one, etc.

The critical thing to remember is that "11" is to be conceived of as one ten and one. The only alteration in counting by ones is that the collection of ten is singled out and held in mind until another collection of ten is counted, then those two collections of ten are held in mind until another collection of ten is counted, etc. This procedure makes it possible to use only the ten digits to write the names for the numbers. Obviously, the written number names for symbols such as "36" must be learned, but it is desirable to learn first that 36 means three tens and six. Since three tens is called thirty, there are thirty and six, or thirty-six objects represented by "36." "Zero" is not to be viewed as a place holder, but rather as the number of elements in the empty set. Thus, in 1032, "0" means that there are no sets of one hundred, just as "3" means that there are three sets of ten -- "0" refers to how many. A critical point in numeration, insofar as learning goes, is when one hundred is reached. To realize its full flavor, consider the numeral "98." The 9 refers to how many tens have been counted and the 8 refers to how many ones have been counted. When one more than 99 is reached, there is no digit to tell how many tens there are, because there are ten tens. So, a new place is created and a new unit for counting is created. The unit is ten tens, which is renamed as one hundred. So, we have one of something, and that something is a hundred. Notationally, "100" is the symbol used where the "1" refers to the number of hundreds. So, when counting beyond 99, we must keep in mind how many hundreds, how many tens, and how many ones.

Children's Conception of Number

From the preceding discussion on number, it ought to be clear that classes (sets) and relations are fundamental to number, both cardinal and ordinal. Because of this close relationship among classes, relations, and number, the preceding material on children's conception of classes and relations is quite pertinent to the discussion on children's conception of number.

One-to-one correspondence is essential to the class usage of cardinal and ordinal number. In the section entitled "Children's Conception of One-To-One Correspondence," three types of one-to-one correspondence were identified from the point of view of the child — essentially no one-to-one correspondence, intuitive one-to-one correspondence, and operational one-to-one correspondence. These three types of one-to-one correspondence determine the quantitative judgments of which children are capable -- gross quantitative judgments, intensive quantitative judgments, and extensive quantitative judgments. The gross quantitative judgments are based on the perceptual features of the situation -- but only one at a time. For example, if two rows of several candies are arranged so one is longer than the other (State 2 in Diagram 18), the child capable only of gross quantitative judgments may say there are more in the longer row, not paying attention to the density of the candies in the two rows, even if they had been previously arranged (State 1 in Diagram 18) so as to be the same length and density. In the case of State 1, the child may have said that the two rows of candies had the same number, but yet, not recognize the contradiction in his judgments about the two states. For the judgments were based on the apparent reality -- the perceptual configurations. Essentially, no one-to-one correspondence exists for the child.



The child capable of intensive quantitative judgments would begin to coordinate the length and density of the objects in the two rows in State 2, but would not yet realize that the increase in length in row 1 is exactly compensated for by a decrease in the density of the objects. He would only admit that if the two rows are put back as they were in State 1, that they would be of the same length.

The child capable of extensive quantitative comparisons would immediately say that the two rows in State 2 are of the same number, because he views the objects as units. For this child, equal number and greater length imples less density. Here, the one-to-one correspondence is operational for the child.

As noted earlier, if the one-to-one correspondence is operational for the child, he should be capable of engaging in transitive reasoning -- the notion of equivalent sets becomes operational. The child, of course, does not know the symbolism involved nor is he aware in any way of set theory. But he is able to reason in concrete situations involving collections of objects.

The child, then, for whom one-to-one correspondence is not operational, would not be capable of the class usage of number, either cardinal or ordinal. On the other hand, the child for whom one-to-one correspondence is operational should be capable of the class usage of number, both cardinal and ordinal. However, the class usage of cardinal and ordinal number involves classification, where the classification is based on one-to-one correspondence (set equivalence). In fact, if a cardinal number, such as 4, is viewed as a particular set, such as {a, b, c, d}, then surely classification is involved even in the member-of-a-class meaning of cardinal number. If {a, b, c, d} is considered as an ordered set, then an asymmetrical, transitive order relation "precedes" is involved in the member-of-a-class usage of ordinal number as well as classification. So, order relations and, hence, seriation is involved in the notion of ordinal number and classification is involved in both cardinal and ordinal number even in the member-of-a-class usage.

It should be clear that if a child is at Stage I (Preoperational) in classification, seriation, or one-to-one correspondence, he doesn't have much chance of dealing with cardinal or ordinal number on any except the most superficial of levels. It must be emphasized that there is nothing

391

"wrong" with a child who is at Stage I on any of the above. All children pass through the stages identified. At Stage II (transitional) in classification, seriation, or one-to-one correspondence, children are beginning to deal with conceptual aspects and definitely are progressing to a more advanced stage of dealing with cardinal and ordinal number. Stage II classification behavior was characterized by children recognizing the intension of a class, but yet, with no coordination between the intension and extension. Children at Stage II are able to partition a class of objects into subclasses, but the subclasses formed are not thought of as forming a hierarchy of classes -- the class inclusion relation is not mastered and hierarchical classification systems are not conceived of. Consequently, there is a good possibility that children at Stage II classification behavior are able to deal with the member-of-a-class meaning of cardinal number in terms of relatively small numbers of elements (less than seven). However, even though a Stage II child is able to cite verbal number names in order, one should not take that to mean that the child is forming successors of sets -- or a sequence of ordinal numbers. His number names constitute a verbal chain, each individual number having a referent, but the numbers are not "nested" for the child because he is not yet capable of forming a hierarchy of classes, which is necessary in dealing with ordinal numbers, and hence, counting.

Stage II classification behavior emerges at about the same time as Stage II seriation behavior, so that the beginning of the member-of-a-class usages of ordinal number is beginning for the child. However, it is not until Stage III that the child comes to a conception of number in its operational sense. He is now capable of classification, seriation, conceives of equivalence and order relations in the sense that his thinking follows these relational patterns. The Stage III child is truly ready for instruction in elementary school mathematics where his already-developed conception of number is refined and extended to include topics normally taught in elementary school mathematics.